Sparsity — tutorial 9

Uniform quasi-wideness, splitter game

Definition 1. Let G be a graph and let $r \in \mathbb{N}$. An *r-quasi-ladder* in G is a pair of sequences $d_1, \ldots, d_\ell$ and $v_1, \ldots, v_\ell$ of vertices in G such that the following conditions hold (see Figure 1):

- $\mathrm{dist}(d_i, v_i) > r$ for all $1 \leq i \leq \ell$; and
- $\mathrm{dist}(d_i, v_j) \leq r$ for all $1 \leq j < i \leq \ell$.

Problem 1. Prove that for every nowhere dense class $\mathcal{C}$ and $r \in \mathbb{N}$, there exists a finite upper bound on the length of r-quasi-ladders that can be found in graphs from $\mathcal{C}$.

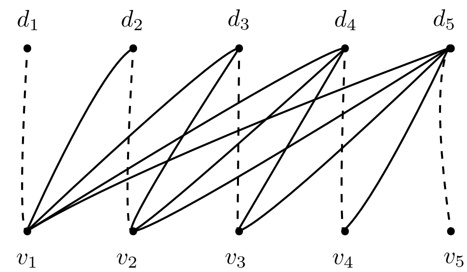

Figure 1: An r-quasi-ladder of length 5. Solid edges depict distance at most r, dashed edges depict distance more than r, no edge means arbitrary relation.

Problem 2. Prove that given $r \in \mathbb{N}$, a graph G, and $A \subseteq V(G)$, the value $\mathrm{dom}_r(G, A)$ can be computed in time $2^{|A|} \cdot |A|^{\mathcal{O}(1)} \cdot (n + m)$.

Problem 3. Consider the following algorithm for the distance-r dominating set problem. The algorithm maintains two sets: $D, S \subset V(G)$. Initially, both are empty, and at each moment, D will have at most k elements. The algorithm proceeds in rounds, each consisting of the *S-step* and then the *D-step*.

> **S-step:** Check whether D r-dominates $V(G)$. If so, terminate and output D as an r-dominating set of size at most k. Otherwise, pick any vertex u which is not r-dominated by D and add it to S.
>
> **D-step:** Check whether some set of at most k vertices r-dominates S. If so, set D to be any such set and proceed to the next round. Otherwise, terminate and conclude that there is no r-dominating set of size at most k.

Let $\mathcal{C}$ be a nowhere dense class and let $r \in \mathbb{N}$. Prove that for every $k \in \mathbb{N}$ there is a constant $L \in \mathbb{N}$, depending only on $k, r, \mathcal{C}$, such that the algorithm terminates after at most L rounds when applied to any $G \in \mathcal{C}$ and k.

Problem 4. Prove that if on a graph G Splitter wins the (ℓ, m, r)-Splitter game, then she also wins the $(\ell \cdot m, 1, r)$-Splitter game.

Problem 5. Prove that for a graph G, the least ℓ for which Splitter wins the $(\ell, 1, \infty)$-Splitter game on G is equal to the treedepth of G.

Problem 6. Let $\mathcal{C}$ be a nowhere dense class. Prove that for all $r, k \in \mathbb{N}$ there exists a constant $M = M(r, k)$ such that the following condition holds. Given $G \in \mathcal{C}$ and its vertex subset $A \subseteq V(G)$ with $|A| > M$, one can compute in polynomial time a vertex $a \in A$ such that A contains an r-independent set of size k if and only if $A - \{a\}$ contains an r-independent set of size k.

Problem 7. Fix a nowhere dense class $\mathcal{C}$ and $r \in \mathbb{N}$. Prove that given $G \in \mathcal{C}$, $A \subseteq V(G)$, and $k \in \mathbb{N}$, one can compute an induced subgraph H of G and $B \subseteq A$ such that the size of H is bounded by a function of k, and in G there is an r-independent set contained in A if and only if in H there is an r-independent set contained in B.

Problem 8. A set family $\mathcal{F}$ is said to have the *k-Helly property* if the following holds. Suppose $\mathcal{G} \subseteq \mathcal{F}$ is a subfamily in which every k-tuple of sets have a nonempty intersection. Then $\mathcal{G}$ has a nonempty intersection. Prove that for each nowhere dense class $\mathcal{C}$ and $r \in \mathbb{N}$, there exists k depending on $\mathcal{C}$ and r such that for every graph $G \in \mathcal{C}$, the family of r-balls $\mathcal{F}_G := \{N_r^G[u] \colon u \in V(G)\}$ has the k-Helly property.