Sparsity — tutorial 8

Uniform quasi-wideness

Definition 1. A class of graph $\mathcal{C}$ is called *uniformly wide* if there is a function $N\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for all $m, r \in N$, $G \in \mathcal{C}$ and $A \subseteq V(G)$ with $|A| \geqslant N(m, r)$ there exists $B \subseteq A$ with $|B| \geqslant m$ such that B is r-independent.

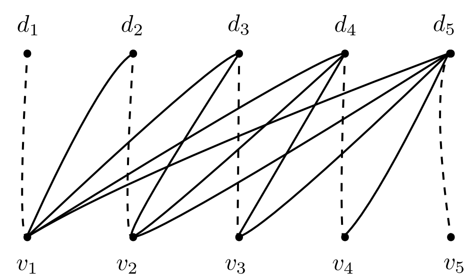
Problem 1. Prove that a class $\mathcal{C}$ of graphs is uniformly wide if and only if $\mathcal{C}$ is a class of bounded degree, i.e., there is a number d such that the maximum degree $\Delta(G)$ of every $G \in \mathcal{C}$ is bounded by d.

Problem 2. Prove that every class of bounded expansion is uniformly quasi-wide.

Definition 2. Let $\mathcal{F}$ be a family of subsets of some universe U. A *chain* in $\mathcal{F}$ is a family $\mathcal{H} \subseteq \mathcal{F}$ such that for all $X, Y \in \mathcal{H}$, we have either $X \subseteq Y$ or $Y \subseteq X$. The *depth* of $\mathcal{F}$ is the cardinality of the longest chain in $\mathcal{F}$. The *intersection closure* of $\mathcal{F}$ is the family of all sets of the form $X_1 \cap X_2 \cap \ldots \cap X_n$ for some $n \in \mathbb{N}$ and $X_1, X_2, \ldots, X_n \in \mathcal{F}$. For $n = 0$ we assume by convention that the intersection of an empty sequence of sets is equal to U, thus the intersection closure always contains the universe U.

Definition 3. Let G be an undirected graph and $r \in \mathbb{N}$. The *r-neighborhood depth* of G, denoted $\mathsf{depth}_r(G)$, is the depth of the intersection closure of the family $\{N^r[u]\colon u \in V(G)\}$. We say that a graph class $\mathcal{C}$ has *bounded neighborhood depth* if there is a function $f\colon \mathbb{N} \to \mathbb{N}$ such that for all $G \in \mathcal{C}$ we have $\mathsf{depth}_r(G) \leqslant f(r)$.

Problem 3. Let G be a graph and $r \in \mathbb{N}$. Show that $\mathsf{depth}_r(G)$ is the largest number $\ell \in \mathbb{N}$ for which there exist vertices $d_1, \ldots, d_{\ell-1}$ and $v_1, \ldots, v_{\ell-1}$ in G such that for all $1 \leqslant j < i < \ell$, d_i does not r-dominate v_i, and d_i r-dominates v_j (see Figure 1).



Rysunek 1: The case $\ell = 6$ in Lemma 3. Solid edges indicate distance at most r, dashed edges indicate distance at least $r + 1$, a lack of an edge allows both possibilities.

Problem 4. Prove that every nowhere dense graph class has bounded neighborhood depth.

Definition 4. For vertices u and v, we say that u *r-dominates* v if their mutual distance is at most r. More generally, a set of vertices A *r-dominates* another set of vertices B if every vertex in B is r-dominated by some vertex of A.

Problem 5. Consider the following algorithm for the distance-r dominating set problem. The algorithm maintains two sets: $D, S \subset V(G)$. Initially, both are empty, and at each moment, D will have at most k elements. The algorithm proceeds in rounds, each consisting of two steps: first the *S-step* and then the *D-step*.

S-step: Check whether D r-dominates $V(G)$. If so, terminate and output D as an r-dominating set of size at most k. Otherwise, pick any vertex u which is not r-dominated by D and add it to S.

D-step: Check whether some set of at most k vertices r-dominates S. If so, set D to be any such set and proceed to the next round. Otherwise, terminate and conclude that there is no r-dominating set of size at most k.

Let $\mathcal{C}$ be a nowhere dense class and let $r \in \mathbb{N}$. Prove that for every $k \in \mathbb{N}$ there is a constant $L \in \mathbb{N}$, depending only on $k, r, \mathcal{C}$ and computable from k for fixed r and $\mathcal{C}$, such that the algorithm terminates after at most L rounds when applied to any $G \in \mathcal{C}$ and k.