# Sparsity — tutorial 7

Tree-width DPs, low tree-depth colorings, neighborhood complexity

**Definition 1.** A *tree-decomposition* of a graph $G$ is a pair $\mathcal{T} \coloneqq (T, \beta)$ consisting of a tree $T$ and a function $\beta\colon V(T) \to 2^{V(G)}$ such that (i) for all $v \in V(G)$ the set $\{t \in V(T) : v \in \beta(t)\}$ is non-empty and induces a connected subtree of $T$; and (i) for every edge $e \in E(G)$ there is $t \in V(T)$ with $e \subseteq \beta(t)$. The *width* of $\mathcal{T}$ is $\max\{|\beta(t)| - 1 : t \in V(T)\}$. The *tree-width* of $G$, denoted $\mathsf{tw}(G)$, is defined as the minimal width of any tree-decomposition of $G$.

**Problem 1.** Let $\mathcal{C}$ be a class of graphs of tree-width at most $k$. Prove that it can be tested in linear time whether $G \in \mathcal{C}$ is 3-colorable.

**Problem 2.** Let $\mathcal{C}$ be a class of graphs of tree-width at most $k$ and let $H$ be graph. Prove that for every $n$-vertex graph $G \in \mathcal{C}$ it can be tested in time $f(|H|, k) \cdot n$ whether $G$ contains a subgraph isomorphic to $H$.

**Problem 3.** Prove that if a graph admits a $p$-tree-depth coloring with $M$ colors, then it also admits a $p$-centered coloring with $M \cdot p^{\binom{M}{<p}}$ colors.

**Definition 2.** Fix $r \in \mathbb{N}$, a graph $G$, and a vertex subset $A \subseteq V(G)$. Let $u \in V(G) - A$ and $a \in A$. A path $P$ connecting $u$ and $a$ is called *$A$-avoiding* if all its vertices apart from $a$ do not belong to $A$. The *$r$-projection profile* of a vertex $u \in V(G) - A$ on $A$ is the function $\mu_r[u, A]\colon A \to \{1, \ldots, r, \infty\}$ defined as follows: for $a \in A$, the value $\mu_r[u, A](a)$ is the length of a shortest $A$-avoiding path connecting $u$ and $a$, or $\infty$ if this length is larger than $r$. A function $f\colon A \to \{1, \ldots, r, \infty\}$ is *realized* as an $r$-projection profile on $A$ if there exists $u \in V(G) - A$ such that $f = \mu_r[u, A]$. The *$r$-projection* of $u$ on $A$ is the set of vertices $a \in A$ reachable from $u$ by an $A$-avoiding path of length at most $r$, i.e., it is the set $(\mu_r[u, A])^{-1}(\{1, \ldots, r\})$.

**Problem 4.** Prove that for every $r \in \mathbb{N}$, graph $G$, subset of its vertices $A \subseteq V(G)$, and $u, v \in V(G) - A$, if $u$ and $v$ have the same $r$-projection profile on $A$ then they also have the same $r$-neighborhood profile on $A$.

**Problem 5.** Prove that for every $r \in \mathbb{N}$ and class $\mathcal{C}$ of bounded expansion there exists a constant $c$, depending only on $\mathcal{C}$ and $r$, such that for every $G \in \mathcal{C}$ and nonempty $A \subseteq V(G)$, the number of different functions from $A$ to $\{1, \ldots, r, \infty\}$ realized as $r$-projection profiles on $A$ is at most $c \cdot |A|$.

In the following exercises we will work out a different proof of the linear bound on the number of different $r$-neighborhoods in classes of bounded expansion. For this, fix $r \in \mathbb{N}$, class $\mathcal{C}$, graph $G \in \mathcal{C}$, and $A \subseteq V(G)$.

**Problem 6.** Reduce the problem to the case where the graph can be partitioned into layers $A = V_0, V_1, \ldots, V_r$, edges in $G$ are only between consecutive layers, and we need to count only the number of different $r$-neighborhoods of the form $N^r_G[u] \cap A$ for $u \in V_r$.

From now on we focus on the case described in the previous problem. Denote $B \coloneqq V_r$, and let us fix some $2r$-centered coloring $\lambda$ of $G$, say with palette $\Gamma$ of size $M$. Consider any path $P$ of length $r$ from $u \in B$ to $v \in A$; such paths will be called *straight paths*. The *signature* of a straight path $P$ is the sequence of colors that consecutive vertices of $P$ receive under $\lambda$. A signature $\sigma \in \Gamma^r$ is *realized* at a vertex $u \in B$ if there is a straight path with signature $\sigma$ starting at $u$.

**Problem 7.** Reduce to the case where all vertices in $B$ realize exactly the same set of signatures.

From now on we adopt this assumption. For $u \in B$ and signature $\sigma \in \Gamma^r$, by $\Pi^\sigma(u)$ we denote the set of those vertices $v \in A$ for which there is a straight path from $u$ to $v$ with signature $\sigma$.

**Problem 8.** Prove that for all $u_1, u_2 \in B$ and $\sigma \in \Gamma^r$, sets $\Pi^\sigma(u_1)$ and $\Pi^\sigma(u_2)$ are either equal or disjoint.

**Problem 9.** Prove that for all $u_1, u_2, u_3 \in B$ and $\sigma_1, \sigma_2 \in \Gamma^r$, it cannot be the case that

$$\Pi^{\sigma_1}(u_1) = \Pi^{\sigma_1}(u_2) \neq \Pi^{\sigma_1}(u_3) \qquad \text{and} \qquad \Pi^{\sigma_2}(u_1) \neq \Pi^{\sigma_2}(u_2) = \Pi^{\sigma_2}(u_3).$$

**Problem 10.** Conclude that the number of different $r$-neighborhoods is linear in the size of $A$.