Sparsity — tutorial 5

Generalized coloring numbers

Problem 1. For all $r, k \in \mathbb{N}$, construct a graph G with $\mathrm{ind}_{2r}(G) = 1$ and $\mathrm{dom}_r(G) \geqslant k$.

Problem 2. Prove that for every graph G and integer $r \in \mathbb{N}$, it holds that $\mathrm{dom}_{2r}(G) \leqslant \mathrm{ind}_r(G)$.

Problem 3. Let $\mathcal{C}$ be a class of bounded expansion and let $r \in \mathbb{N}$. Give a two-line proof that there exists a constant $c_r \in \mathbb{N}$ such that every $2r$-independent set I in a graph $G \in \mathcal{C}$ contains a $(2r+1)$-independent set I' of size at least $|I|/c_r$. How would a similar argument work if one only assumed that $\mathcal{C}$ is nowhere dense?

Problem 4. Suppose H is a bipartite graph with bipartition (X, Y) which satisfies the following conditions: vertices of Y have pairwise different neighborhoods in X, and every vertex of Y has at least one neighbor in X. Prove that there exists a mapping $\phi \colon Y \to X$ with the following properties:

- for each $y \in Y$, we have $\phi(y)y \in E(H)$; and
- for each $x \in X$, we have $|\phi^{-1}(X)| \leqslant \mathrm{wcol}_2(G) + 2^{\mathrm{wcol}_2(G)}$.

Infer the qualitative result from the previous tutorial, about the linear bound on the number of 1-neighborhoods in bounded expansion classes.

Problem 5. Suppose we are given a graph G with its vertex ordering σ satisfying $\mathrm{wcol}_r(G, \sigma) \leqslant c$, for some constants c and r. Suppose further that every vertex u is also supplied with lists of vertices of $\mathrm{WReach}_{r'}[G, \sigma, u]$, for each $r' \leqslant r$. Prove that given such a data structure, it is possible to answer the following query in constant time: for given vertices u, v of G, is it true that $\mathrm{dist}(u, v) \leqslant r$?

Problem 6. Let $\mathcal{C}$ be a class of bounded expansion and let $r, k \in \mathbb{N}$. Prove that given a graph $G \in \mathcal{C}$ one can construct a data structure that can answer the following queries: for given vertices $u_1, \ldots, u_k$ of G, is it true that there exists a vertex v satisfying $\mathrm{dist}(u_i, v) \leqslant r$ for all $i \in \{1, 2, \ldots, k\}$. The data structure should have the following parameters:

- The running time of constructing the data structure is $\mathcal{O}(n^c)$ for some universal constant c, independent of $\mathcal{C}$, r, and k.
- The running time of answering one query is $\mathcal{O}(1)$.

Constants hidden in the $\mathcal{O}(\cdot)$-notation may depend on $\mathcal{C}$, r, and k.

Problem 7. Let G be a graph that excludes K_t as a minor, for some $t \in \mathbb{N}$. Consider the following procedure for computing a vertex ordering σ of G. At each step, we maintain a family of *blobs* $B_1, B_2, \ldots, B_p \subseteq V(G)$, which are pairwise disjoint, and we let $R \coloneqq V(G) - \bigcup_{i=1}^{p} B_i$ be the set of vertices which are not yet in any blob. To create the next blob, we take C to be any connected component of $G[R]$, and let u be any vertex of C. Create blob B_{p+1} as follows: start with $\{u\}$, and for every blob B_i that is adjacent to C, pick any vertex $v \in C$ adjacent to B_i, and add to B_{p+1} any shortest path from u to v within C. Finally, when all vertices are already in blobs, create a vertex ordering σ by ordering blobs according to the order of their creation, and ordering vertices within each blob arbitrarily.

Prove that the ordering σ defined in this manner satisfies $\mathrm{col}_r(G, \sigma) \leqslant \mathcal{O}(rt^2)$ for all $r \in \mathbb{N}$.