Sparsity — tutorial 3

Measuring sparsity

Fact 1. If G is a graph and $r, c \in \mathbb{N}$, then

$$\nabla_r(G \bullet K_c) \leqslant 2c^2(r+1)^2\nabla_r(G) + c.$$

Problem 1. Prove that if a class $\mathcal{C}$ is nowhere dense, then for every constant $c \in \mathbb{N}$ the class $\mathcal{C} \bullet K_c$ is also nowhere dense.

Problem 2. Suppose $\mathcal{C}$ is a class of bounded expansion. Prove that for every $r \in \mathbb{N}$ there exists a constant c_r such that the following holds. For every graph $G \in \mathcal{C}$ and every its vertex subset $A \subseteq V(G)$, there exists a vertex subset $B \supseteq A$ with the following properties:

- $|B| \leqslant c_r|A|$, and
- for every pair of vertices $u, v \in A$, if $\mathrm{dist}_G(u, v) \leqslant r$ then $\mathrm{dist}_{G[B]}(u, v) = \mathrm{dist}_G(u, v)$.

Problem 3. Prove that a d-degenerate n-vertex graph has at most $2^d \cdot n$ cliques.

Problem 4. Suppose G is a graph and $A \subseteq V(G)$ some subset of its vertices. Define the following equivalence relation $\sim_A$ on the vertices of $V(G) - A$:

$$u \sim_A v \qquad \text{if and only} \qquad N[u] \cap A = N[v] \cap A.$$

Prove that

- in $V(G) - A$, the number of vertices with at least $2\nabla_0(G)$ neighbors in A is at most $|A|$; and
- $\sim_A$ has at most $(4^{\nabla_1(G)} + \nabla_1(G)) \cdot |A|$ equivalence classes.

Problem 5. Let G be a graph.

- Suppose that G has $2m$ edges. Show that there exists a bipartite subgraph $H \subseteq G$ of G with m edges.
- Suppose G has $n > 10^9$ vertices. Use the probabilistic method to prove that there exists a subgraph $H \subseteq G$ of G with $m/4$ edges and vertex partition $V(H) = A \cup B$ such that $||A| - n/2| \leqslant n/100$.
- Assume G has $2n$ vertices and $2m$ edges. Show that there exists a bipartite subgraph $H \subseteq G$ of G with m edges and vertex partition $V(H) = A \cup B$ such that $|A| = |B|$.