Sparsity — tutorial 2

Measuring sparsity

Problem 1. Design an algorithm that given a graph G on n vertices and m edges, computes the degeneracy and the optimum degeneracy ordering of G in time $\mathcal{O}(n+m)$.

Problem 2. Prove that every graph on 2^{a+b} vertices contains either a clique of size a or an independent set of size b. Conclude that there is a function $N(p,k)$ such that every complete graph on $N(p,k)$ vertices with edges colored with p colors contains a monochromatic clique on k vertices.

Problem 3. Prove that the following statements for a graph class $\mathcal{C}$ are equivalent:

- The class $\mathcal{C}$ is somewhere dense.
- There is $r \in \mathbb{N}$ s.t. for every $t \in \mathbb{N}$, the exact r-subdivision of K_t is a subgraph of some graph from $\mathcal{C}$.

Definition 1. For graphs G, H, their *lexicographic product* $G \bullet H$ is defined as the graph on the vertex set $V(G) \times V(H)$ where (u,v) and (u',v') are adjacent if either $u \neq u'$ and $uu' \in E(G)$, or $u = u'$ and $vv' \in E(H)$. The *c-blowup* of a graph G is the graph $G \bullet K_c$.

Problem 4. Prove that if G is a graph and $r, c \in \mathbb{N}$, then

$$\widetilde{\nabla}_r(G \bullet K_c) \leqslant 2rc^3 \cdot \widetilde{\nabla}_r(G) + c^2.$$

Infer that there for each r there is a polynomial $P_r(\cdot,\cdot)$ such that

$$\nabla_r(G \bullet K_c) \leqslant P_r(c, \nabla_r(G)).$$

Conclude that if a class $\mathcal{C}$ has bounded expansion, then for every constant $c \in \mathbb{N}$ the class $\mathcal{C} \bullet K_c = \{G \bullet K_c \colon G \in \mathcal{C}\}$ also has bounded expansion.

Fact 1. If G is a graph and $r, c \in \mathbb{N}$, then

$$\nabla_r(G \bullet K_c) \leqslant 2c^2(r+1)^2 \nabla_r(G) + c.$$

Problem 5. Prove that if a class $\mathcal{C}$ is nowhere dense, then for every constant $c \in \mathbb{N}$ the class $\mathcal{C} \bullet K_c$ is also nowhere dense.

Problem 6. Suppose $\mathcal{C}$ is a class of bounded expansion. Prove that for every $r \in \mathbb{N}$ there exists a constant c_r such that the following holds. For every graph $G \in \mathcal{C}$ and every its vertex subset $A \subseteq V(G)$, there exists a vertex subset $B \supseteq A$ with the following properties:

- $|B| \leqslant c_r|A|$, and
- for every pair of vertices $u, v \in A$, if $\mathrm{dist}_G(u,v) \leqslant r$ then $\mathrm{dist}_{G[B]}(u,v) = \mathrm{dist}_G(u,v)$.

Problem 7. Prove that a d-degenerate n-vertex graph has at most $2^d \cdot n$ cliques.

Problem 8. Suppose G is a graph and $A \subseteq V(G)$ some subset of its vertices. Define the following equivalence relation $\sim_A$ on the vertices of $V(G) - A$:

$$u \sim_A v \qquad \text{if and only} \qquad N[u] \cap A = N[v] \cap A.$$

Prove that

- in $V(G) - A$, the number of vertices with at least $2\nabla_0(G)$ neighbors in A is at most $|A|$; and
- $\sim_A$ has at most $4^{\nabla_1(G)} + \nabla_1(G)$ equivalence classes.