Sparsity — tutorial 15

To the infinite

Problem 1. Describe the following ultraproducts:

- For a fixed graph H, describe $\prod_{\mathcal{U}} H$.
- Describe $\prod_{\mathcal{U}} K_n$.

Conclude that $\mathcal{C} \subseteq \mathcal{C}^\star$ for every class $\mathcal{C}$.

Problem 2. Prove that if $\mathcal{C}$ is somewhere dense, then $K_{2^\omega} \in \mathcal{C}^\star \nabla \omega$.

Problem 3. Prove that if $\mathcal{C}$ has bounded degree, then $\mathcal{C}^\star$ also has bounded degree. Prove that if $\mathcal{C}$ has bounded treedepth, then $\mathcal{C}^\star$ also has bounded treedepth.

Problem 4. Prove that if $\mathcal{C}$ is topologically nowhere dense, then $\mathcal{C}^\star$ is limit topologically nowhere dense.

Problem 5. Prove the infinite Ramsey's theorem: If G is a graph on infinitely many vertices, then G contains either an infinite clique or an infinite independent set.

Problem 6. Infer the finite Ramsey's theorem from the infinite one using the method of ultraproducts.

Problem 7. Suppose H is a bipartite graph with sides of the bipartition A and B. Suppose further that A is infinite and every vertex in B has a finite degree. Prove that one of the following conditions holds:

- A contains an infinite subset of vertices that pairwise have no common neighbors; or
- H admits K_ω as a 1-shallow minor.

Definition 1. Class $\mathcal{C}$ is *limit uniformly quasi-wide* if for every $r \in \omega$, every graph $G \in \mathcal{C}$, and infinite subset of vertices $A \subseteq V(G)$, there exists a finite set $S \subseteq V(G)$ and an infinite set $B \subseteq A - S$ that is r-independent in $G - S$.

Problem 8. Prove that every class of graphs is limit nowhere dense class if and only if it is limit uniformly quasi-wide.