Sparsity — tutorial 14

Polynomial expansion and approximation algorithms

Problem 1. In this problem we will work out a proof of the following statement: if $\mathcal{F}$ is a (k, t)-packing in G, then for each r we have

$$\nabla_r(G[\mathcal{F}]) \leq \frac{k-1}{2} + (2(k-1)(2rt + r + t + 1) + 1) \cdot \nabla_{2rt+t+r}(G). \tag{1}$$

(a) Let H be a depth-r minor of $G[\mathcal{F}]$. For $v \in V(H)$, let I_h be the subgraph of G induced by the union of elements $\mathcal{F}$ contained in the branch set of h. Prove that I_h has radius at most $2rt + r + t$. For every $h \in V(H)$ fix a center $c(h)$ as above.

(b) Prove that there are at most $\frac{k-1}{2} \cdot |V(H)|$ edges hh' in H with $c(h) = c(h')$.

(c) Prove that for $hh' \in E(H)$ with $c(h) \neq c(h')$ there is a path $P_{hh'}$ connecting $c(h)$ and $c(h')$ and composed of two subpaths of length at most $2rt + r + t$ each: one contained in I_h and second contained in $I_{h'}$.

(d) Call edges $hh', gg' \in E(H)$ with distinct endpoints *conflicting* if $P_{hh'}$ and $P_{gg'}$ intersect. Similarly, say that $hh', gg' \in E(H)$ with $h = g$ are *conflicting* if they intersect outside of the subpaths contained in $I_h = I_g$. Prove that if $F \subseteq E(H)$ is a subset of pairwise non-conflicting edges, then the subgraph of H consisting of edges in F and vertices incident on them is a depth-$(2rt + r + t)$ minor of G.

(e) Prove that every edge of H is in conflict with at most $(k - 1)(4rt + 2r + 2t + 2)$ other edges.

(f) Show that the number of edges $hh' \in E(H)$ with $c(h) \neq c(h')$ is $\leq (2(k-1)(2rt+r+t+1)+1) \cdot \nabla_{2rt+t+r}(G)$.

(g) Conclude (1).

Problem 2. In this exercise we will work out EPTASes for several problems on planar graphs. Let us start with the Independent Set problem.

(a) Suppose the graph is connected and fix any optimum solution I. Pick any vertex $v \in V(G)$ and run BFS from v. Let $p = \lceil 1/\epsilon \rceil$. Prove that there exists $i \in \{0, 1, \ldots, p - 1\}$ such that layers congruent to i modulo p in total contain at most $\epsilon|I|$ elements of I.

(b) Prove that the removal of layers as above yields a graph of treewidth at most $3p$.

(c) Show that in an n-vertex planar graph, a $(1 - \epsilon)$-approximate independent set can be found in time $\mathcal{O}(2^{3/\epsilon} \cdot n)$.

(d) Use a similar reasoning to obtain EPTASes for the following problems on planar graphs: Dominating Set, r-Independent Set, r-Dominating Set.

Problem 3. Consider the (c, r)-Dominating Set problem, which is a variant of standard r-Dominating Set where every vertex has to be r-dominated by c different elements of the solution. Let $\mathcal{C}$ be a class of polynomial expansion and $c, r \in \mathbb{N}$ be fixed. Prove that for each $\epsilon > 0$ there exists $\lambda \in \mathbb{N}$, depending only on $\mathcal{C}, c, r, \epsilon$, such that λ-local search yields a PTAS on $\mathcal{C}$ for (c, r)-Dominating Set.

Problem 4. A graph class $\mathcal{C}$ is *weakly hyperfinite* if for every $\epsilon > 0$ there exists $C \in \mathbb{N}$ such that from every n-vertex graph $G \in \mathcal{C}$ one may delete at most ϵn vertices so that each connected component of the remaining graph has size at most C. Prove that every graph class of polynomial expansion is weakly hyperfinite.

Problem 5. Using a black-box result from previous lectures, prove that given an r-dominating set L, the existence of a λ-close r-dominating set of size smaller than L can be decided in time $f(\lambda) \cdot n$, for some computable function f. Conclude that on any class of polynomial expansion $\mathcal{C}$ and any $\epsilon > 0$, there is a $(1 + \epsilon)$-approximation algorithm for r-Dominating Set on $\mathcal{C}$ with running time $f(\lambda) \cdot n^2$.