Sparsity — tutorial 11

Stability and VC dimension

Problem 1. A graph G is a *map graph* if with every vertex u of G we may associate a polygon R_u in the plane so that the interiors of regions R_u are pairwise disjoint and two vertices $u, v \in V(G)$ are adjacent in G if and only if the corresponding regions R_u and R_v share a point on their boundaries. Prove that the class of map graphs has bounded order dimension, but is not nowhere dense.

In the solution of the next problem you may use the following fact.

Fact 1. Let G be an n-vertex graph such that $K_t \not\preccurlyeq_1 G$ for some constant $t \in \mathbb{N}$. Then G has at most $\binom{t+1}{2} \cdot n^{3/2}$ edges.

Problem 2. Let G be a graph such that $K_t \not\preccurlyeq_4 G$ for some $t \in \mathbb{N}$ and let $A \subseteq V(G)$. Suppose that every pair of elements of A has a common neighbor in $V(G) - A$. Prove that there exists a vertex v in $V(G) - A$ which has $\Omega_t(|A|^{1/8})$ neighbors in A.

In the next problems we will prove the following fact.

Fact 2. Let G be a bipartite graph with partitions A and B. Let $m, t, d \in \mathbb{N}$ and suppose $|A| \geq (m + d)^{2t}$. Then at least one of the following assertions hold:

(a) A contains a 2-independent set of size m;

(b) $K_t \preccurlyeq_4 G$;

(c) B contains a vertex of degree at least $\Omega_t(d^{1/8})$.

For the proof of Fact 2, construct a binary tree labelled with all the elements of A, by inserting them one by one and using the following question when comparing the inserted vertex $v \in A$ with a vertex $u \in A$ already residing in the tree: do u and v have a common neighbor? If the answer is positive, proceed to the son of the current node, otherwise proceed to the daughter of the current node. Let τ be the obtained tree.

Problem 3. Prove that if τ has a node with alternation $2t$ then $K_t \preccurlyeq_2 G$.

Problem 4. Prove that if τ has a node with m letters D, then A contains a 2-independent set of size m.

Problem 5. Prove that if τ has a node with d letters S, then either $K_t \preccurlyeq_4 G$ or B contains a vertex of degree at least $\Omega_t(d^{1/8})$.

Problem 6. Prove Fact 2 and discuss how it can be used to show that the function $N(r, m)$ in uniform quasi-wideness is actually a polynomial in m (with degree depending on the class and r).

Problem 7. Let $\mathcal{C}$ be a nowhere dense class of graphs and let $G \in \mathcal{C}$. For $r \in \mathbb{N}$ and a vertex ordering σ of G, let

$$\mathcal{W}_{r,\sigma} = \{\mathrm{WReach}_r[G, \sigma, v] \;:\; v \in V(G)\}.$$

Prove that there exists a constant k, depending only on $\mathcal{C}$ and r (and not on G and σ), such that $\mathcal{W}_{r,\sigma}$ has VC-dimension at most k.

Problem 8. Let $\mathcal{F}$ be a set system over a finite ground set A. We define the *dual* set system $\mathcal{F}^\star$ as follows: the ground set of $\mathcal{F}^\star$ is $\mathcal{F}$, and for each $e \in A$ we add to $\mathcal{F}^\star$ the set $\{X \in \mathcal{F} \colon e \in X\}$. Prove that if the VC dimension of $\mathcal{F}$ is k, then the VC dimension of $\mathcal{F}^\star$ is at most $2^{k+1} - 1$.