Parameterized algorithms — tutorial 9

Cut problems

Let us recall definition of *important cut*. Let G be an undirected graph and let $X, Y \subseteq V(G)$ be two disjoint sets of vertices. Let $S \subseteq E(G)$ be an (X, Y)-cut and let R be the set of vertices reachable from X in $G \setminus S$. We say that S is an important (X, Y)-cut if it is inclusion-wise minimal and there is no (X, Y)-cut S' with $|S'| \leqslant |S|$ such that $R \subset R'$, where R' is the set of vertices reachable from X in $G \setminus S'$.

Problem 1. Prove that bound 4^k for number of important (X, Y)-cuts of size at most k is optimal up to polynomial factor i.e. show an example where we have at least $4^k k^{-O(1)}$ important (X, Y)-cuts of size at most k.

Problem 2. In Edge Multiway Cut we are given a graph G, set of terminals $T = \{t_1, \ldots, t_{|T|}\}$ and a nonnegative integer k and we are asked whether it is possible to find set $S \subseteq E(G)$ so that $|S| \leqslant k$ and every connected component of $G \setminus S$ contains at most one terminal.
Prove that we can solve Edge Multiway Cut problem on trees in polynomial time.

Problem 3. In Edge Multicut problem we are given a graph G, pairs of vertices $(s_1, t_1), \ldots, (s_l, t_l)$ and a nonnegative interger k and we are asked whether there is a set $S \subseteq E(G)$ so that $|S| \leqslant k$ and for every $1 \leqslant i \leqslant l$ there is no path between s_i and t_i in $G \setminus S$.
Prove that we can solve Edge Multicut problem on trees in $\mathcal{O}^\star(2^k)$ time.

In (p, q)-Partition problem we are given a graph G and integers p and q and we are asked whether there exists partition of $V(G)$ into disjoint sets $V_1, \ldots, V_k$ (called *clusters*) so that for every $1 \leqslant i \leqslant k$ we have that $|V_i| \leqslant p$ and $d(V_i) \leqslant q$ where $d(X)$ for $X \subseteq V(G)$ is number of edges with exactly one end in X.
In exercises 4-8 we are going to prove that (p, q)-Partition is FPT when parametrized by q.

Problem 4. Let $f : 2^{V(G)} \to \mathbb{R}$ be called *posimodular* iff for every $A, B \subseteq V(G)$ it satisfies $f(A) + f(B) \geqslant f(A \setminus B) + f(B \setminus A)$. Prove that function $d(X)$ is posimodular.

Problem 5. For (p, q)-Partition problem to have solution there is an obvious necessary condition that every vertex has to be in some (p, q)-cluster. Prove that this condition is sufficient as well. Deduce that this problem is solvable in $n^{O(q)}$ time.

Problem 6. In Satellite Problem we are given a graph G, integers p, q, a vertex $v \in V(G)$ and a partition $(V_0, V_1, \ldots, V_r)$ of $V(G)$ such that $v \in V_0$ and there is no edge between V_i and V_j for any $1 \leqslant i < j \leqslant r$ (note that it's not „$0 \leqslant i < j \leqslant r$"). The task is to find a (p, q)-cluster C satisfying $V_0 \subseteq C$ such that for every $1 \leqslant i \leqslant r$, either $C \cap V_i = \emptyset$ or $V_i \subseteq C$. Prove that this problem can be solved in polynomial time.

For a given graph G and its vertex v we say that a set $X \subseteq V(G)$ is *important* if:

- $d(X) \leqslant q$,
- $G[X]$ is connected,
- there is no $Y \supset X, v \notin Y$ such that $d(Y) \leqslant d(X)$ and $G[Y]$ is connected.

Problem 7. Let C be an inclusion-wise minimal (p, q)-cluster containing v. Prove that every component of $G \setminus C$ is an important set.

Problem 8. Given an n-vertex graph G, vertex $v \in V(G)$, and integers p and q, we can construct in time $2^{O(q)} n^{O(1)}$ an instance I of the Satellite Problem such that

- If some (p, q)-cluster contains v, then I is a yes-instance with probability $2^{-O(q)}$,
- If there is no (p, q)-cluster containing v, then I is a no-instance.

Conclude we can solve (p, q)-Partition in $\mathcal{O}^{\star}(2^{O(q)})$.