Parameterized algorithms — tutorial 5

Treewidth

Problem 1. Prove that if $H \preccurlyeq G$ then $tw(H) \leqslant tw(G)$. In other words, prove that for every fixed k class of graphs with treewidth bounded by k is closed under taking minors.

Problem 2. Prove that if X is a clique in a graph G then in every tree decomposition T of G there is a bag containing all vertices of X.

Problem 3. A graph G is $d-$*degenerate* if and only if in every subgraph of G there is a vertex that has degree at most d. Prove that every graph of treewidth at most d is d-degenerate.
Note that it follows that graphs of bounded treewidth have linear number of edges. Moreover graphs of treewidth d are $d+1$-colorable.

Problem 4. We are given a graph G so that $tw(G) \leqslant k$ and $|G| = n$. Design algorithms solving following problems in $O(f(k)n)$ time:

1. 3-COLORABILITY
2. ODD CYCLE TRANSVERSAL
3. DOMINATING SET
4. FEEDBACK VERTEX SET
5. H-MINOR TESTING for fixed H
6. H-MINOR HITTING for fixed H

Problem 5. We are given graph $G = (V, E)$. Provide $\mathbf{MSO}_2$ formula expressing that:

1. $X \subseteq V$ is an independent set
2. $X \subseteq V$ is a vertex cover
3. $X \subseteq V$ is a dominating set
4. $X \subseteq V$ hits every H-minor for fixed H

Conclude that corresponding problems are FPT when parameterized by treewidth.

Problem 6. By using facts connected with treewidth, prove that FEEDBACK VERTEX SET is FPT (parameterized by size of solution).

Theorem 1. (Robertson-Seymour) For every class of graphs $\mathcal{C}$ which is closed under taking minors there is finite set of graphs S such that $G \in \mathcal{C} \Leftrightarrow \forall_{H \in S} \not\preccurlyeq G$.

Problem 7. We are given a graph G and integer k and some fixed integer η. In η-transversal problem we are asked whether there exists $X \subseteq V(G)$ such that $|X| \leqslant k$ and $G \setminus X$ has treewidth at most η. Prove that this problem is nonuniform FPT when parameterized by k.