Parameterized algorithms — tutorial 4

Randomization

In all problems algorithms are allowed to be randomized unless explicitly stated otherwise.

Problem 1. In Tree Subgraph Isomorphism problem we are given a tree H that has k vertices and a graph G and we are asked whether G has a subgraph isomorphic to H. Prove that this problem can be solved in $\mathcal{O}^\star(c^k)$ time for some constant c.
If by any chance you are familiar with treewidth notion you can solve generalized problem for patterns from class of graphs with bounded treewidth instead of for just trees.

Problem 2. In Partial Vertex Cover problem we are given graph G and integers k and t and we are asked whether there exists a set $X \subseteq V(G)$ of size at most k such that at least t edges have at least one end in X. Prove that we can solve this problem in $\mathcal{O}^\star(c^t)$ time for some constant c.
You can give a try to analogous problem — Partial Dominating Set.

Problem 3. For positive integers c, d we define *$c-$stack of $d-$onions* as a graph with vertices $\{v_i : 0 \leqslant i \leqslant c\}$ and $\{x_{i,j} : 1 \leqslant i \leqslant c, 1 \leqslant j \leqslant d\}$ where every vertex $x_{i,j}$ has degree two and is connected to v_{i-1} and v_i. Let's denote $k = 1 + c + cd$ — total number of vertices in *$c-$stack of $d-$onions*. Prove that we can check whether a given graph G contains *$c-$stack of $d-$onions* as a subgraph in $\mathcal{O}^\star(s^k)$ time for some constant s.

Problem 4. We are given universe U, integers k and t and a family $\mathcal{F} \subseteq 2^U$ so that for every $A \in \mathcal{F}$ we have $|A| \leqslant d$ and we are asked whether there exists a set $X \subseteq V(G)$ so that for at least t sets $A \in \mathcal{F}$ we have $|X \cap A| = 1$. Prove that this problem can be solved in $\mathcal{O}^\star((de)^t)$ time.

Problem 5. We are given a graph G and integers q and k. We call a partition $V(G) = A \uplus B$ a *$(q, k)-$good cut* if $G[A]$ and $G[B]$ are connected, $|A|, |B| \geqslant q$ and there are at most k edges between A and B. Prove that we can check whether G admits (q, k)-*good cut* in:

1. randomized $\mathcal{O}^\star(2^{O(q+k)})$ time
2. randomized $\mathcal{O}^\star(2^{O(k \log q)})$ time
3. deterministic $\mathcal{O}^\star(2^{O(q+k)})$ time
4. deterministic $\mathcal{O}^\star(2^{O(k \log q)})$ time

Problem 6. Show that any $(n, 2, 2)$-splitter has $\Omega(\log n)$ elements.
Note: In some way it says that $\log n$ *factor is unavoidable.*

Problem 7. We are given a graph G so that every vertex has degree at most 3 and an integer k. Our goal is to find a subgraph of size k that has the biggest possible number of edges. Prove that we can solve this problem in $\mathcal{O}^\star(c^k)$ time for some constant c.