Parameterized algorithms — tutorial 2

Kernelization

Problem 1. Consider following problem: we are given a set $\mathcal{A}$ of points on a plane and an integer k and we are asked whether there exist k lines so that every point in $\mathcal{A}$ belongs to at least one of these lines. Prove that we can reduce this instance into instance $(\mathcal{A}', k')$ where $k' \leqslant k$ and $|\mathcal{A}'| \leqslant k^2$.

Problem 2. We are given a set P of politicians, a set B of businessmen and an information about which politicians know which businessmen. Relationships between pairs of politicians and pairs of businessmen are irrelevant. By wiretapping a specific person we can eavesdrop all conversations between this person and all of his friends. In our problem we are asked wheter we can eavesdrop all pairs of friends by wiretapping at most p politicians and b businessmen. Prove that this problem admits a kernel with at most $2pb$ people.

Problem 3. In Feedback Arc Set in Tournaments problem we are given a tournament (directed clique) and asked whether it is possible to reverse at most k edges to make it acyclic. Prove that this problem admits a kernel with $O(k^2)$ vertices.

Problem 4. In Cluster Editing problem we are given a graph G, positive integer k and we are asked whether it is possible to change (add or remove) at most k edges in order to get a graph where every connected component is a clique. Prove that this problem admits a kernel with $O(k^2)$ vertices.

Problem 5. In Set Cover problem we are given set U, family $\mathcal{F}$ of subsets of U and an integer k and we are asked whether there exists a subfamily $\mathcal{F}' \subseteq \mathcal{F}$ of size at most k so that $\bigcup \mathcal{F}' = U$. In this problem we consider an instance of Set Cover problem where every pair of two sets $A, B \in \mathcal{F}$ fulfills a condition $|A \cap B| \leqslant d$ for some constant d. Prove that with this condition Set Cover admits a kernel of polynomial size when parameterized by k.
Comment: Note that since d is constant, it can appear in an exponent of kernel's size.

Problem 6. In $d-$Set Packing problem we are given an integer k, set U and family $\mathcal{F} \subseteq 2^U$, so that every element of $\mathcal{F}$ has size at most d and we are asked whether there exist k pairwise disjoint sets in $\mathcal{F}$. Prove that this problem admits a kernel with at most $d!(dk)^d$ sets.

Problem 7. In Max-Sat problem we are given CNF-SAT formula and an integer k and we are asked whether there exist a truth assignment that satisfies at least k clauses.

Show that this problem admits a kernel with at most $2k$ clauses and with:

1. $O(k^2)$ variables
2. $O(k)$ variables

Problem 8. We will say that graph H is *c-almost-clique* if every vertex of H is not a neighbour of at most c other vertices at H (or in other words, if max degree of complement of H is at most c). We consider a problem of removing at most k vertices out of G so that every connected component of resulting graph is *c-almost-clique*. Show that this problem admits a polynomial kernel for every constant c when parameterized by k. If you are brave then show that this problem admits polynomial kernel when parameterized by $k + c$.