Parameterized algorithms — tutorial 13

Exponential Time Hypothesis and Strong Exponential Time Hypothesis

For an integer q the $q-SAT$ problem is a version of $CNF-SAT$ problem where size of every clause is restricted to be at most q. Let δ_q be the infimum of the set of constants c for which there exists a deterministic algorithm solving $q-SAT$ in time complexity $\mathcal{O}^\star(2^{cn})$.

Conjecture 1. Exponential Time Hypothesis (ETH): $\delta_3 > 0$

Conjecture 2. Strong Exponential Time Hypothesis (SETH): $\lim_{q\to\infty} \delta_q = 1$

Problem 1. Prove that SETH implies ETH.

Problem 2. Prove that Dominating Set can't be solved in time $2^{o(n+m)}$ unless ETH fails (where n and m denote number of vertices and number of edges in our graph respectively).

Problem 3. Prove that 3-Coloring can't be solved in time $2^{o(n+m)}$ unless ETH fails.

Interlude about planar problems and algorithms in time $2^{O(\sqrt{n+m})}$.

Problem 4. In $k \times k$ Clique problem we are given a graph whose vertex set is $[k] \times [k]$ ($[k] = \{1, 2, \ldots, k\}$) which we will refer to as a $k \times k$ table where vertex (i, j) is vertex in $i-$th row and $j-$th column. We are asked whether it contains a clique on k vertices so that it contains one vertex from each row. Prove that this problem does not admit an algorithm in time $2^{o(k \log k)}$ unless ETH fails.
Hint: Try reducing some known hard graph problem to this problem. Divide graph into small parts and let every row denote partial solutions on corresponding part of graph.

Problem 5. In $k \times k$ Hitting Set problem we are given a graph whose vertex set is $[k] \times [k]$ and a family $\mathcal{F}$ of its subsets and we are asked whether there exists a set F of k vertices so that it contains one vertex from each row and every set from $\mathcal{F}$ contains at least one element of F. Prove that this problem does not admit an algorithm in time $2^{o(k \log k)}$ unless ETH fails.

In $k \times k$ Hitting Set With Thin Sets we additionally require that sets in $\mathcal{F}$ contain at most one vertex from each row. This problem does not admit an algorithm in time $2^{o(k \log k)}$ unless ETH fails as well, however it is harder to prove.

Problem 6. In Closest String problem we are given a set of n strings of length m over alphabet Σ and an integer d and we are asked whether exists a string S so that it differs on at most d positions from every string from our set. Prove that this problem does not admit an algorithm in time $2^{o(d \log d)}$.