Parameterized algorithms — tutorial 12

W[1] and W[2]-hardness

Problem P is $W[1]$-hard when parametrized by k if we can find an FPT-reduction (with parameter k) from Clique to P.

Problem P is $W[2]$-hard when parametrized by k if we can find an FPT-reduction from Dominating Set to P.

Problem 1. In Induced Matching problem we are given a graph G, an integer k and we are asked whether there exist $2k$ vertices $x_1, \ldots, x_k$ and $y_1, \ldots y_k$ which induce matching, or in other words so that there are no edges between x_i and x_j, no edges between y_i and y_j and there is an edge between x_i and y_j if and only if $i = j$. Prove that this problem is $W[1]$-hard when parameterized by k.

Problem 2. In Dominating Clique problem we are given a graph G, an integer k and we are asked whether there exists a set $X \subseteq V(G)$ such that $|X| = k$, $G[X]$ is a clique and $N[X] = V(G)$. Prove that this problem is $W[2]$-hard when parameterized by k.

In exercises 3-5 we will investigate Dominating Set on Tournaments problem. In this problem we are given a tournament T (directed graph so that there is an edge between every pair of vertices) and an integer k and we are asked whether there exists a set $X \subseteq V(T)$ such that $|X| \leqslant k$ and for every $v \in T$ either $v \in X$ or there exists $u \in X$ such that there is an edge (u, v).

Problem 3. Prove that every tournament on n vertices has dominating set of size at most $1 + \log_2(n)$. Conclude that we can solve Dominating Set in Tournament in quasi-polynomial time (it means $O(n^{polylog(n)})$).

Problem 4. Prove that for every $k \in \mathbb{N}$ there exists a tournament such that it doesn't have dominating set of size at most k. Provide an algorithm that computes example of such tournament in $f(k)$ time.
Hint: Use probabilistic method.

Problem 5. Prove that there exists FPT-reduction from Set Cover to Dominating Set on Tournaments.

Problem 6. In Grid Tiling problem we are given an integer k, an integer n and k^2 sets of pairs $S_{i,j} \subseteq [n] \times [n]$ for $1 \leqslant i, j \leqslant k$. The task is to find for each $1 \leqslant i, j \leqslant k$ a pair $s_{i,j} \in S_{i,j}$ such that for each valid choice of i and j we have that $s_{i,j}$ and $s_{i,j+1}$ agree on first coordinate $s_{i,j}$ and $s_{i+1,j}$ agree on second coordinate. Prove that this problem is $W[1]$-hard when parameterized by k.

Problem 7. In Unit Disk Independent Set we are given set on disks of diameter 1 on a plane and an integer k and we are asked whether there exist k pairwise disjoint disks. In other words we need to choose k centers so that distance between every pair of them is more than 1. Prove that this problem is $W[1]$-hard when parameterized by k.