Parameterized algorithms — tutorial 11

Representative sets

Let A be a family of sets of size p. A subfamily $\mathcal{A}' \subseteq \mathcal{A}$ is said to q-represent $\mathcal{A}$ if for every set B of size q such that there is an $A \in \mathcal{A}$ that is disjoint from B, there is an $A' \in \mathcal{A}'$ that also is disjoint from B. If $\mathcal{A}'$ q-represents $\mathcal{A}$, we write $\mathcal{A}' \subseteq_q^{rep} \mathcal{A}$.

Problem 1. For every p and q, show an example of a p-family $\mathcal{A}$ so that there is no $\mathcal{A}' \subseteq \mathcal{A}$ of size smaller than $\binom{p+q}{p}$ that q-represents $\mathcal{A}$.

Problem 2. In d-Hitting Set problem we are given integer k, universe U and a family $\mathcal{F}$ of subsets of universe that are of size at most d and we are asked whether there exists a set H such that $\forall_{F \in \mathcal{F}} H \cap F \neq \emptyset$ and $|H| \leqslant k$. In Ed-Hitting Set problem we additionally require that sets in $\mathcal{F}$ have size *exactly* d. By using representative sets show that:

1. Ed-Hitting Set admits a kernel with at most $\binom{k+d}{d}$ sets
2. d-Hitting Set admits a kernel with at most $\binom{k+d}{d}$ sets

Problem 3. In d-Set Packing problem we are given integer k, universe U and a family $\mathcal{F}$ of subsets of universe that are of size at most d and we are asked whether $\mathcal{F}$ contains k disjoint sets. In Ed-Set Packing problem we additionally require that sets in $\mathcal{F}$ have size *exactly* d. By using representative sets show that:

1. Ed-Set Packing admits a kernel with at most $\binom{kd}{d}$ sets
2. d-Set Packing admits a kernel with at most $\binom{kd}{d}$ sets

Throwback An (n, k)-*universal set* is a family $\mathcal{U}$ of subsets of $[n]$ such that for any $S \subseteq [n]$ of size k, the family $\{A \cap S : A \in \mathcal{U}\}$ contains all 2^k subsets of S.
For any $n, k \geqslant 1$ one can construct (n, k)-universal set of size $2^k k^{O(\log k)} \log n$ in time $2^k k^{O(\log k)} n \log n$.

Problem 4. Give an algorithm that, given as input a family $\mathcal{A}$ of sets of size p over a universe of size n, computes a q-representative family $\mathcal{A}'$ of $\mathcal{A}$. The algorithm should have running time at most $|\mathcal{A}| \cdot 2^{p+q+o(p+q)} n^{O(1)}$ and the size of $\mathcal{A}'$ should be at most $2^{p+q+o(p+q)} \log n$. Can you shave off $\log n$ factor from size of $\mathcal{A}'$ (while not increasing running time significantly)?