MAG — exercise session 9

Treedepth, mso and fo

Definition 1. The *treedepth* of a graph G is the minimum height of a rooted forest F with $V(F) = V(G)$ such that G is contained in the ancestor-descendant closure of F.

Problem 1. Prove the following recursive formula for treedepth. If G is disconnected, then $\mathsf{td}(G)$ is equal to the maximum of $\mathsf{td}(G_i)$ for G_i ranging over the connected components of G. Otherwise, if G is connected, then $\mathsf{td}(G)$ is equal to the minimum of $1 + \mathsf{td}(G - v)$ for v ranging over the vertices of G.

Problem 2. Prove that $\mathsf{td}(P_n) \geqslant 1 + \log_2 n$, where P_n is the path on n vertices. Conclude that a subgraph-closed graph class $\mathcal{C}$ has bounded treedepth if and only if there is a universal bound on the diameters of connected components of graphs from $\mathcal{C}$.

Problem 3. Prove that if H is a minor of G, then $\mathsf{td}(H) \leqslant \mathsf{td}(G)$.

Problem 4. Prove that for every positive integer d, there exists a finite list $\mathcal{L}_d$ of graphs such that $\mathsf{td}(G) \leqslant d$ if and only if G does not contain any member of $\mathcal{L}_d$ as an induced subgraph. Conclude that there is an fo sentence τ_d that checks whether a given graph G has treedepth at most d.

Problem 5. Let G be a connected graph of treedepth at most d. Prove that the number of vertices v such that $\mathsf{td}(G - v) < \mathsf{td}(G)$ is bounded by $f(d)$, for some function $f(\cdot)$. Conclude that such vertices can be selected by an fo formula $\psi_d(v)$ with one free variable.

Problem 6. Consider the following decomposition algorithm, run on a graph of treedepth at most d. If the graph is disconnected, then partition it into connected components and decompose each of them in parallel. Otherwise, remove from the graph all vertices v such that the removal of v decreases the treedepth. Prove that this algorithm results in a treedepth decomposition of depth at most $g(d)$ for some function $g(\cdot)$. Prove that for each $i \leqslant d$, there exists an fo formula $\alpha_i(v)$ that selects vertices removed in the ith step of the procedure.

Problem 7. Fix integers s, d, and let Σ_s be the signature of undirected graphs colored with s unary predicates and with a treedepth decomposition given by a binary ancestor-descendant relation. Prove that for each mso sentence φ over Σ_s there exists an fo sentence $\overline{\varphi}$ over Σ_s such that for all c-colored graphs G enriched with a treedepth decomposition of depth at most d, it holds that $G \models \varphi$ if and only if $G \models \overline{\varphi}$.

Problem 8. Fix integer d. Conclude that for each mso sentence φ on graphs there exists an fo sentence $\overline{\varphi}$ on graphs such that for all graphs G of treedepth at most d, it holds that $G \models \varphi$ if and only if $G \models \overline{\varphi}$.

Problem 9. Give an mso sentence ψ on graphs with the following property. For every graph class $\mathcal{C}$ that is closed under taking subgraphs and has unbounded treedepth, there is no fo sentence on graphs which selects the same set of graphs from $\mathcal{C}$ as ψ.

Problem 10. Conclude the following theorem of Elberfeld, Grohe, and Tantau. If $\mathcal{C}$ is closed under taking subgraphs, then mso collapses to fo on $\mathcal{C}$ if and only if $\mathcal{C}$ has bounded treedepth.