MAG — exercise session 3

Courcelle's theorem, mso transductions

Problem 1. Prove that for every integer k, the class of graphs of treewidth at most k is mso_2-definable.

Problem 2. We are given term t over the algebra $\mathbb{A}_k$ of k-interface graphs and a positive integer q. Suppose t evaluates to a k-interface graph $G\langle t\rangle$. Show how to compute the mso-type of rank q of $G\langle t\rangle$ in time $f(q,k)\cdot n$, where n is the size of t and f is computable.

Problem 3. We are given a graph G and an mso_2 formula $\varphi(X)$ with one free monadic vertex variable X. The question is to find the minimum cardinality of X such that $G, X \models \varphi$. Prove that this can be done in time $f(k, \|\varphi\|)\cdot n$ for some computable f, where k is the treewidth of G. What if we want to (a) count the number of such sets X, or (b) determine the existence of such a set X of cardinality exactly p, for a given p?

Problem 4. Prove that HAMILTONICITY cannot be expressed in mso_1.

Problem 5. Give an mso_2 transduction that given a graph, outputs its decomposition into 2-connected components.

Problem 6. Prove that every mso transduction can be expressed in the normal form:

$$\mathcal{I} = \mathcal{I}_{\text{rename}} \circ \mathcal{I}_{\text{restrict}} \circ \mathcal{I}_{\text{interprete}} \circ \mathcal{I}_{\text{copy}} \circ \mathcal{I}_{\text{filter}} \circ \mathcal{I}_{\text{color}},$$

where the above are mso transductions as follows:

- $\mathcal{I}_{\text{color}}$ is a finite sequence of coloring steps;
- $\mathcal{I}_{\text{filtering}}$ is a single filtering step;
- $\mathcal{I}_{\text{copy}}$ is a single copying step;
- $\mathcal{I}_{\text{interprete}}$ is a finite sequence of interpretation steps;
- $\mathcal{I}_{\text{restrict}}$ is a single universe restriction step;
- $\mathcal{I}_{\text{rename}}$ is a single renaming step.

Moreover, such normal form can be computed given $\mathcal{I}$ as a sequence of atomic transductions.

Fact 1. Given an mso formula $\varphi(x_1, x_2, \ldots, x_r)$ and a relational structure of treewidth k, one can in time $f(\|\varphi\|, k)\cdot(n+m)$ output all tuples $(a_1, \ldots, a_r)$ that satisfy φ in G, where n and m are sizes of input and output, respectively.

Problem 7. Prove that given an mso transduction $\mathcal{I}$ and a structure $\mathfrak{A}$ of treewidth k, one can in time $f(\|\mathcal{I}\|, k)\cdot(n+m)$ output either any element of $\mathcal{I}(\mathfrak{A})$, or conclude that it is empty. Here, n and m are sizes of input and output, respectively.