MAG — exercise session 11

FII, protrusions, meta-kernelization

Definition 1. For a subset property Π, the problem $\min\langle\Pi\rangle$ is *separable* if there exists a function $f(p)$ such that for any two p-interface graphs $\mathbb{G}$ and $\mathbb{H}$ we have the following. If S is any optimum solution for $\min\langle\Pi\rangle$ on $\mathbb{G} \oplus \mathbb{H}$, then

$$|\mathsf{OPT}(\mathbb{G}) - |S \cap V(\mathbb{G})|| \leqslant f(p).$$

If $f(p)$ is linear, then Π is *linear separable.*

Definition 2. A subset property Π has *finite integer index* if for every p, the following equivalence relation $\sim_\Pi$ on p-interface graphs has finite index. Two p-interface graphs $\mathbb{G}_1$ and $\mathbb{G}_2$ are considered equivalent if and only if there exists an integer c such that for every p-interface graph $\mathbb{H}$ we have $\mathsf{OPT}(\mathbb{G}_1 \oplus \mathbb{H}) - \mathsf{OPT}(\mathbb{G}_2 \oplus \mathbb{H}) = c$.

Definition 3. Suppose Π is mso-definable by a formula $\phi(X)$. Fix p and define an equivalence relation $\equiv_\phi$ on p-interface graphs with vertex subsets as $(\mathbb{G}_1, S_1) \equiv_\phi (\mathbb{G}_2, S_2)$ if and only if for all $(\mathbb{H}, T)$ it holds that $\mathbb{G}_1 \oplus \mathbb{H} \models \phi(S_1 \cup T)$ iff $\mathbb{G}_2 \oplus \mathbb{H} \models \phi(S_2 \cup T)$. Note that $\equiv_\phi$ has finite index.

Definition 4. Assume Π is mso-definable by a formula $\phi(X)$ and is separable with function $f(p)$. For a p-interface graph $\mathbb{G}$ and an equivalence class κ of $\equiv_\phi$, define $\mathsf{OPT}(\mathbb{G}, \kappa)$ to be the smallest size of S such that $(\mathbb{G}, S) \in \kappa$, or $+\infty$ if such S does not exist. Further define $\gamma_{\mathbb{G}}(\kappa)$ as $\mathsf{OPT}(\mathbb{G}, \kappa) - \mathsf{OPT}(\mathbb{G})$ if the absolute value of this number is at most $f(p)$, or $\perp$ otherwise.

Problem 1. In the setting of the above definition, suppose p-interface graphs $\mathbb{G}_1, \mathbb{G}_2$ satisfy $\gamma_{\mathbb{G}_1}(\kappa) = \gamma_{\mathbb{G}_2}(\kappa)$ for all equivalence classes κ of $\equiv_\phi$; call this common function $\gamma(\cdot)$. Take any p-interface graph $\mathbb{H}$ and let S_1, S_2 be optimum solutions in $\mathbb{G}_1 \oplus \mathbb{H}$ and $\mathbb{G}_2 \oplus \mathbb{H}$, respectively. Suppose further that $(\mathbb{G}_1, S_1 \cap V(\mathbb{G}_1)) \in \kappa_1$ and $(\mathbb{G}_2, S_2 \cap V(\mathbb{G}_2)) \in \kappa_2$ for κ_1, κ_2 being equivalence classes of $\equiv_\phi$.

(1) Prove that $|S_1 \cap V(\mathbb{G}_1)| - \mathsf{OPT}(\mathbb{G}_1) = \gamma(\kappa_1)$ and $|S_2 \cap V(\mathbb{G}_2)| - \mathsf{OPT}(\mathbb{G}_2) = \gamma(\kappa_2)$.

(2) Prove that $\mathbb{G}_1 \oplus \mathbb{H}$ admits a solution of size $\gamma(\kappa_2) + \mathsf{OPT}(\mathbb{G}_1) + |S_2 \setminus V(\mathbb{G}_2)|$, while an optimum solution in $\mathbb{G}_1 \oplus \mathbb{H}$ has size $\gamma(\kappa_1) + \mathsf{OPT}(\mathbb{G}_1) + |S_1 \setminus V(\mathbb{G}_1)|$.

(3) Prove that $\mathbb{G}_2 \oplus \mathbb{H}$ admits a solution of size $\gamma(\kappa_1) + \mathsf{OPT}(\mathbb{G}_2) + |S_1 \setminus V(\mathbb{G}_1)|$, while an optimum solution in $\mathbb{G}_2 \oplus \mathbb{H}$ has size $\gamma(\kappa_2) + \mathsf{OPT}(\mathbb{G}_2) + |S_2 \setminus V(\mathbb{G}_2)|$.

(4) Conclude that $\mathsf{OPT}(\mathbb{G}_1 \oplus \mathbb{H}) - \mathsf{OPT}(\mathbb{G}_2 \oplus \mathbb{H}) = \mathsf{OPT}(\mathbb{G}_1) - \mathsf{OPT}(\mathbb{G}_2)$.

Conclude that every separable and mso-definable subset minimization problem has FII.

Definition 5. A *τ-protrusion* in a graph G is a set of vertices X such that $G[X]$ has treewidth at most τ and there are at most τ vertices in X that have neighbors outside of X. The set of latter vertices is called the *boundary of* X, and denoted by ∂X.

Problem 2. Suppose a graph G contains a τ-protrusion X. Prove that then for any $c \leqslant |X|$, G has a τ-protrusion of size between c and $2c$.

Definition 6. An *η-transversal* in a graph G is a set of vertices Z such that $G - Z$ has treewidth at most η.

Problem 3. Prove that if a planar graph G admits an η-transversal of size k, then G has treewidth at most $\alpha\sqrt{k}$ for some constant α depending on η.

Problem 4. Let $C > 100$ and η be fixed, and let α be the constant given for η by the previous exercise. Denote

$$\rho = \frac{1+\sqrt{2}+\sqrt{3}}{\sqrt{3}} \qquad \text{and} \qquad \delta = \frac{2C\alpha}{\rho}.$$

Moreover, choose k_0 depending only on α so that for every $k \geqslant k_0$ we have

$$\frac{2}{3}k + \alpha\sqrt{k} \leqslant k - 1 \qquad \text{and} \qquad \frac{k}{3} - \frac{2\alpha}{\rho}\sqrt{\frac{k}{3}} \geqslant 1.$$

Define $\tau = \max(\eta, k_0)$. Prove the following statement: every graph G with η-transversal Z of size $k > 1$ and such that $|V(G) \setminus Z| \geqslant Ck$ admits a τ-protrusion with interior of size at least C. Do this by verifying the statement directly for $k \leqslant k_0$, and proving by induction that for $k \geqslant \frac{1}{3}k_0$ a stronger statement holds, where we assume only $|V(G) \setminus Z| \geqslant Ck - \delta\sqrt{k}$.

Problem 5. Prove that if a connected graph G admits a dominating set of size k, then it admits a connected dominating set of size at most $3k$.

Problem 6. Prove that if D is a connected dominating set in a planar graph G, then the treewidth of $G - D$ is bounded by some universal constant. Conclude that if a planar graph G admits a dominating set of size k, then G admits an η-transversal of size at most $3k$, for some universal constant η.

Definition 7. The problem $\min\langle\Pi\rangle$ is *minor-bidimensional* if the following conditions hold:

- If H is a minor of G, then $\mathsf{OPT}(H) \leqslant \mathsf{OPT}(G)$.
- There exist constants $\delta > 0$ and c such that $\mathsf{OPT}(\mathrm{Grid}_{k\times k}) \geqslant \delta k^2 - c$ for all k.

Problem 7. Prove that η-Transversal is minor-bidimensional, for each η.

Problem 8. Suppose Π is a subset problem such that $(G, S) \in \Pi$ depends only on whether the graph $G - S$ belongs to some fixed graph class $\mathcal{C}$. Prove that if Π is minor-bidimensional, then $\mathcal{C}$ has bounded treewidth.