MAG — exercise session 10

FII, protrusions, bidimensionality

Definition 1. A *subset property* is a set Π of pairs (G, S), where $S \subseteq V(G)$. The property Π is mso-*definable* if there is an mso-formula $\varphi(X)$ such that $G \models \phi(S)$ if and only if $(G, S) \in \Pi$. By $\min\langle\Pi\rangle$ we denote the following problem: given G, compute $\mathsf{OPT}_\Pi(G) = \min\{|S| : (G, S) \in \Pi\}$.

Definition 2. For a graph property Π, the problem $\min\langle\Pi\rangle$ is *separable* if there exists a function $f(p)$ such that for any two p-interface graphs $\mathbb{G}$ and $\mathbb{H}$ we have the following. If S is any optimum solution for $\min\langle\Pi\rangle$ on $\mathbb{G} \oplus \mathbb{H}$, then

$$|\mathsf{OPT}(\mathbb{G}) - |S \cap V(\mathbb{G})|| \leqslant f(p).$$

If $f(p)$ is linear, then Π is *linear separable.*

Problem 1. Prove that the following condition is sufficient to ensure that the property Π is linear separable. For any p and any two p-interface graphs $\mathbb{G}$ and $\mathbb{H}$, the following implications hold:

- For any $S \subseteq V(\mathbb{G} \oplus \mathbb{H})$, if $(\mathbb{G} \oplus \mathbb{H}, S) \in \Pi$ then $(\mathbb{G}, (S \cap V(\mathbb{G})) \cup \mathrm{int}(\mathbb{G})) \in \Pi$.
- For any $S \subseteq V(\mathbb{G}), T \subseteq V(\mathbb{H})$, if $(\mathbb{G}, S), (\mathbb{H}, T) \in \Pi$ then $(\mathbb{G} \oplus \mathbb{H}, S \cup T \cup \mathrm{int}(\mathbb{G}) \cup \mathrm{int}(\mathbb{H})) \in \Pi$.

Problem 2. Conclude that the following problems are linear separable: Vertex Deletion to H-minor-free (for connected H), r-Dominating Set, Connected Dominating Set.

Definition 3. A graph property Π has *finite integer index* if for every p, the following equivalence relation $\sim_\Pi$ on p-interface graphs has finite index. Two p-interface graphs $\mathbb{G}_1$ and $\mathbb{G}_2$ are considered equivalent if and only if there exists an integer c such that for every p-interface graph $\mathbb{H}$ we have $\mathsf{OPT}(\mathbb{G}_1 \oplus \mathbb{H}) - \mathsf{OPT}(\mathbb{G}_2 \oplus \mathbb{H}) = c$.

Problem 3. Suppose Π is mso-definable by a formula $\phi(X)$. Fix p and define an equivalence relation $\equiv_\phi$ on p-interface graphs with vertex subsets as $(\mathbb{G}_1, S_1) \equiv_\phi (\mathbb{G}_2, S_2)$ if and only if for all $(\mathbb{H}, T)$ it holds that $\mathbb{G}_1 \oplus \mathbb{H} \models \phi(S_1 \cup T)$ iff $\mathbb{G}_2 \oplus \mathbb{H} \models \phi(S_2 \cup T)$. Prove that $\equiv_\phi$ has finite index.

Problem 4. In the setting from the exercise above, assume in addition that Π is separable with function $f(p)$. For a p-interface graph $\mathbb{G}$ and an equivalence class κ of $\equiv_\phi$, let define $\mathsf{OPT}(\mathbb{G}, \kappa)$ to be the smallest size of S such that $(\mathbb{G}, S) \in \kappa$, or $+\infty$ if such S does not exist. Further define $\gamma_{\mathbb{G}}(\kappa)$ as $\mathsf{OPT}(\mathbb{G}, \kappa) - \mathsf{OPT}(\mathbb{G})$ if the absolute value of this number is at most $f(p)$, or $\perp$ otherwise. Prove that if p-interface graphs $\mathbb{G}_1, \mathbb{G}_2$ satisfy $\gamma_{\mathbb{G}_1}(\kappa) = \gamma_{\mathbb{G}_2}(\kappa)$ for all equivalence classes κ of $\equiv_\phi$, then $\mathbb{G}_1 \sim_\Pi \mathbb{G}_2$. Conclude that every separable and mso-definable subset minimization problem has finite integer index.

Definition 4. A *t-protrusion* in a graph G is a set of vertices X such that $G[X]$ has treewidth at most t and there are at most t vertices in X that have neighbors outside of X. The set of latter vertices is called the *boundary of* X, and denoted by ∂X.

Problem 5. Suppose a graph G contains a t-protrusion X. Prove that then for any $c \leqslant |X|$, G has a t-protrusion of size between c and $2c$.

Problem 6. Prove that for every fixed t, there is a polynomial-time algorithm (with degree possibly depending on t) that given a graph G and integer c, either concludes that there is no t-protrusion of size at least c, or finds a $2t$-protrusion of size at least c.

Definition 5. The problem $\min\langle\Pi\rangle$ is *minor-bidimensional* if the following conditions hold:

- If H is a minor of G, then $\mathsf{OPT}(H) \leqslant \mathsf{OPT}(G)$.
- There exist constants $\delta > 0$ and c such that $\mathsf{OPT}(\mathrm{Grid}_{k\times k}) \geqslant \delta k^2 - c$ for all k.

Problem 7. Suppose Π is a subset problem such that $(G, S) \in \Pi$ depends only on whether the graph $G - S$ belongs to some fixed graph class $\mathcal{C}$. Prove that if Π is minor-bidimensional, then $\mathcal{C}$ has bounded treewidth.

Problem 8. Suppose $\min\langle\Pi\rangle$ is minor-bidimensional and linear-separable and G is a planar graph with $\mathsf{OPT}(G) = k$. Prove that there is a partition A, X, B of the vertex set of G with no edges between A and B such that $|X| \leqslant \mathcal{O}(\sqrt{k})$, and $\mathsf{OPT}(G[A \cup X]), \mathsf{OPT}(G[B \cup X]) \leqslant \frac{2}{3}k + \mathcal{O}(\sqrt{k})$.

Problem 9. Prove that if $\min\langle\Pi\rangle$ is minor-bidimensional and linear-separable, then there exist constants c, η such that every planar graph G with $\mathsf{OPT}(G) = k$ admits a vertex subset X of size at most $\mathcal{O}(k^c)$ such that $G - X$ has treewidth at most η.

Note: During next lecture we will make a refined analysis showing that one can take $c = 1$.

Problem 10. Let G be a planar graph and let A be a subset of its vertices. Prove the following:

- There are at most $2|A|$ vertices of $V(G) \setminus A$ that have more than 2 neighbors in A.
- Vertices of $V(G) \setminus A$ with at most 2 neighbors in A may be partitioned into at most $4|A|$ classes so that vertices in the same class have exactly the same neighbors in A.