Commutative Algebra Fall 2014/2015

Problem set XI, for January13th
More fields.

Relevant readings: Aluffi, Chapter VII of *Algebra, Chapter 0* and Milne *Fields and Galois theory*. By $\mathbb{F}_p$ be denote the field of p elements.

1. Separable closure. Consider an extension $K \subseteq L$. Prove that the set of elements of L which are separable over K is a field. We call it separable closure of K in L. Consider the extension $K = \mathbb{F}_p(x^p) \subset L = \mathbb{F}_p(x)$ and find the separable closure of K in L.

2. In this exercise we consider the extension $\mathbb{F}_p(x^p, y^p) \subset \mathbb{F}_p(x, y)$.

   (a) Prove that this extension is finite and normal.

   (b) Find the separable closure of $\mathbb{F}_p(x^p, y^p)$ in $\mathbb{F}_p(x, y)$.

   (c) Prove that this extension has infinite number of intermediate extensions.

3. We consider a field extension $K \subseteq L$. A differentiation of K with values in L is a function $D : K \to L$, additive $D(a + b) = D(a) + D(b)$ and satisfying Leibnitz rule $D(ab) = dD(b) + bD(a)$.

   (a) Prove that for every $a \in K$ we have $D(a^n) = na^{n-1}D(a)$ and for $f \in K[x]$ it holds $D(f(a)) = f^D(a) + f'(a) \cdot D(a)$, where f' is the usual differential and f^D is obtained from f by replacing its coefficients by their differentials with respect to D.

   (b) Suppose that D_1 and D_2 are differentiations, then also $D_1 + D_2$ and $D_1D_2 - D_2D_1$ are differentiations.

   (c) Prove that any perfect field of positive characteristic has only zero differentiations.

4. Let $K_1 \subset K_2$ be a finite separable extension. Prove that any differentiation D of K_1 can be uniquely extended to a differentiation $\overline{D}$ of K_2. Hint: note that $K_2 = K_1(a)$ where a is separable with minimal polynomial $f_a \in K_1[x]$; show $\overline{D}(a) = -f_a^D(a)/f_a'(a)$.

5. Let $K \subset L$ be a finite extension. For $a \in L$ the multiplication by a defines K-linear homomorphism $L \to L$. By $Tr_{L/K}(a)$ and $N_{L/K}(a)$ we denote the trace and determinant of this homomorphism. This determines maps from L to K which we denote by $Tr_{L/K}$ and $N_{L/K}$, respectively, and call the trace and norm of the extension.

   (a) Suppose $L = K(a) \supset K$ and that $f = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ is the minimal polynomial for a. Find $Tr_{L/K}(a)$ and $N_{L/K}(a)$.

   (b) Prove that $Tr_{L/K}$ is K-linear on L and $N_{L/K}$ is a homomorphism of multiplicative groups $L^* \to K^*$

   (c) Prove that the composition of finite extensions $K \subset L \subset M$; yields a good composition of their traces: $Tr_{L/K} \circ Tr_{M/L} = Tr_{M/K}$.

   (d) For $K \subset L$ as above we define $T : L \times L \to K$ for $a,\ b \in L$ setting $T(a, b) = Tr_{L/K}(ab)$. Prove T is K-bilinear and symmetric.

6. Let us assume that $K \subset L$ is a finite separable extension. By $\varphi_1, \ldots \varphi_m$ we denote all possible embeddings of L in $\overline{K}$ (fixed on K).

   (a) Prove that $Tr_{L/K}(a) = \sum_i \varphi_i(a)$ and $N_{L/K}(a) = \prod_i \varphi_i(a)$.

   (b) Prove that T is non-degenerate (of maximal rank).

7. Let $\mu_d \in \mathbb{C}$ denote the primitive root of the unit of degree d.

   (a) Prove that the extension $\mathbb{Q} \subset \mathbb{Q}(\mu_d)$ is Galois.

   (b) Prove that $Aut(\mathbb{Q}(\mu_d))$ is commutative and, if moreover d is prime, then it is cyclic.

   (c) Find $Aut(\mathbb{Q}(\mu_d))$ for $d = 4,\ 5,\ 6$ and find the intermediate extensions of $\mathbb{Q} \subset \mathbb{Q}(\mu_d)$ associated to the respective subgroups of $Aut(\mathbb{Q}(\mu_d))$.