Commutative Algebra Fall 2014/2015

Problem set X, for December 16th
Introduction to fields.

The text for this part of the lectures is Aluffi, Chapter VII of *Algebra, Chapter 0* and Milne, *Fields and Galois theory.*

1. Let $\overline{K}$ denote algebraic closure of a field K. Prove that every automorphism of K extends to an automorphism of $\overline{K}$.

   (a) Prove that $\overline{\mathbb{Q}}$ has infinite number of automorphisms.

   (b) Prove that the group of automorphims of $\mathbb{R}$ is trivial. Hint: first show that if $x > 0$ and $h \in Aut(\mathbb{R})$ then $h(x) > 0$.

2. Let K be a field of characteristic $p > 0$. We define a Frobenius map $\Phi = \Phi_K : K \to K$ by setting $\Phi(a) = a^p$.

   (a) Show that Φ is an endomorphism of K.

   (b) Prove that if K is finite or algebraically closed then Φ is an automorphism.

   (c) Prove that $\Phi_{K(x)}$ is not an automorphism of the field of rational functions $K(x)$.

   (d) Find the field of invariants of Φ^n, that is K^{Φ^n}.

   (e) Prove that for every n every algebraically closed field of characterisitic p contains exactly one subfield of cardinality p^n.

   (f) Prove: if the cardinality of K is p^n then its group of automorphims is cyclic of cardinality n and generated by Φ.

3. A field K is called perfect if every algebraic extension of K is separable.

   (a) Prove that every finite field is perfect.

   (b) Prove that a field of characteristic $p > 0$ is perfect if and only if its Frobenius endomorphism is an automorphism.

   (c) Prove that K of characteristic $p > 0$ is perfect if and only if for every $a \in K$ the polynomial $x^p - a$ has a root in K.

4. Recall that for a polynomial $f \in K[x]$ its field of decomposition is the smallest algebraic extension $K_f \supseteq K$ which contains all roots of f.

   (a) Prove that $K_f \subseteq \overline{K}$ is the intersection of fields in which f decomposes into linear factors.

   (b) Prove that the extension $K \subseteq K_f$ is normal.

   (c) Prove that if $K \subset L$ is a normal and finite extension then $L = K_f$ for some $f \in K[x]$.

   (d) Suppose that $\mathrm{char} K \neq 2$ and $K \subset L$ is an extension of degree 2. Prove that there exits $a \in K$ such that L is the field of decomposition of $x^2 - a$.

   (e) Let $\deg f = d$; prove that $[K_f : K] \leq d\,!$