Commutative Algebra Fall 2014/2015

Problem set IX, for December 9th
Group actions on polynomial rings and their invariants.
More about group actions on the plane and their rings of invariants can be found in notes of Miles Reid; follow links regarding cyclic quotient singularities and Du Val singularities.

1. Multiplicative characters of the group. A character of a (finite or finitely generated) group G is a homomorphism $\chi : G \to \mathbb{C}^*$, where $\mathbb{C}^*$ is the multiplicative group of complex numbers.

   (a) Prove that the set of characters is a group, we denote it by $G^\vee$.

   (b) Let G be finite and abelian; prove that $G^\vee$ is isomorphic to G.

2. Cyclic group action. Let $G_m \subset \mathbb{C}^*$ be the multiplicative group of m-th roots of unity generated by $\epsilon = \epsilon_m = \exp(2\pi i/m)$. Let us assume that G_m acts linearly on $\mathbb{C}$-vector space $V = \mathbb{C}^n$, that is we have a monomorphism $G_m \to GL(V)$.

   (a) Prove that the action is diagonalizable, that is there exists a basis $(v_1, \ldots, v_n)$ of V and integers $0 \leq a_i < m$ such that $\epsilon(v_i) = \epsilon^{a_i} \cdot v_i$; we call a_i's the weights of the action.

   (b) We say that the action of a group is faithful if no element except the unit acts as the identity. Prove that the action of G is faithful if and only if $(m, a_1, \ldots, a_n) = 1$.

   (c) Let $(x_1, \ldots, x_n)$ be the coordinates on V in which G_m acts linearly. We extend the action of G_m to the action on the ring of polynomials $\mathbb{C}[x_1, \ldots, x_n]$. Prove that monomials $x_1^{b_1} \cdots x_n^{b_n}$ are eigenvectors of this action. The eigenvalue of the action associated to the monomial is called its weight.

   (d) Identify monomials in $\mathbb{C}[x_1, \ldots, x_n]$ with the monoid $M_{\geq 0} = \mathbb{Z}_{\geq 0})^0$. We extend $M_{\geq 0}$ to the lattice $M \simeq \mathbb{Z}^n$. Prove that associating the weight of a monomial determines a homomorphism $M \to G_m^\vee \simeq \mathbb{Z}_m$.

3. Invariants of the cyclic group action. Notation as in the previous exercise.

(a) Prove that $x_1^{b_1} \cdots x_n^{b_n} \in \mathbb{C}[x_1, \dots, x_n]^{G_m}$ if and only if the exponent $(b_1, \dots, b_n) \in M$ is in the kernel of the above homomorphism.

(b) Without using Hilbert theorem, prove that the algebra of invariants $\mathbb{C}[x_1, \dots, x_n]^{G_m}$ is generated by a finite number of monomials. Bound the number of generators in terms of n and m.

(c) Find the condition when $\mathbb{C}[x_1, \dots, x_n]^{G_m}$ has $n+1$ generators.

(d) Find generators of $\mathbb{C}[x_1, x_2]^{G_m}$ for $(a_1, a_2) = (a, m-a)$, with $0 < a < m$. Find relations between these generators.

4. Let us consider the binary dihedral group $BD_{4m} \subset SL(2, \mathbb{C})$ generated by

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \epsilon^{-1} & 0 \\ 0 & \epsilon \end{pmatrix}$$

with $\epsilon = exp(2\pi i/2m)$ the $2m$-th root of unity. Find the rank of the group and the relation between the generators. The group BD_{4m} acts linearly on $\mathbb{C}[x_1, x_2]$. Find the generators of the ring of invariants and relations between them. Use the previous excercise.

5. Find all invariant rings of complex polynomials in one variable $\mathbb{C}[x]$. Prove first that any finite group acts on $\mathbb{C}$ via a character which generates a finite group of roots of unity; prove that any such ring is isomorphic to $\mathbb{C}[x]$.

6. Let us assume that the ring $A = \mathbb{C}[x_1, \dots, x_n]$ admits a linear action of a finite group G. For any $\chi \in G^\vee$ we define

$$A^G_\chi = \{a \in A \mid \ \forall_{g \in G} \ \ g(a) = \chi(g) \cdot a\}$$

(a) Show that A^G_χ is a finite A^G module.

(b) Show that if G is cyclic (more generally: abelian) then then we have na isomorphism of A^G-modules $A = \bigoplus_{\chi \in G^*} A^G_\chi$.