Commutative Algebra Fall 2014/2015

Problem set VIII, for November 25th
Ideals in polynomial rings, symmetric polynomials.

The material regarding these topics can be found in chapter 2 of Cox, Little, O'Shea book. In particular the definitions of orders *lex*, *grlex* and *grevlex* on $\mathbb{Z}_{\geq 0}$ can be found there. You may want to do some computer supported calculations. I suggest you register at the http://sage.mimuw.edu.pl and use our local sage serves to do the calculations. You may browse the worksheets posted by other people; you can start with the simplest one about Groebner basis, division algorithm. I encourage you to post your own worksheets.

Recall that a finite set of polynomials $f_1, \dots f_r$ in and ideal I of $k[x_1, \dots x_n]$ is called a Gröbner basis if their leading terms generate the ideal of leading terms of I, that is $(LT(f_1), \dots, LT(f_r)) = LT(I)$. By S_n we understand the group of permutations of n elements.

1. We know that the natural order on $\mathbb{Z}_{\geq 0}$ has the following property: for any two elements $\alpha > \beta$ there exists only a finite number of γ's which satisfy $\alpha > \gamma > \beta$. Which of the orders *lex*, *grlex* and *grevlex* on $\mathbb{Z}^n_{\geq 0}$ has this property?

2. We consider a polynomial $f = xy^2z^2 + xy - yz$ and a list of polynomials $F = [x - y^2, y - z^3, z^2 - 1]$ in $k[x, y, z]$. Use the division algorithm to find the remainder of f with respect to F for orders *lex* and *grlex*. Next change the order in F, that is consider $F' = (z^2 - 1, y - z^3, x - y^2)$. Hint: use *sage* or any other algebra program.

3. Show that the division algorithm is k-linear. That is, if for $i = 1,\ 2$ the remainder of f_i by division by F is r_i then for $c_i \in k$ the remainder of $c_1f_1 + c_2f_2$ on division by F is $c_1r_1 + c_2r_2$.

4. Let G and G' be two Gröbner basis of an ideal I in a polynomial ring with a fixed order. Prove that the remainders of the division algorithm by G and G' are the same.

5. Let G be a Gröbner basis of an ideal I. Prove that a polynomial f is in I if and only if the remainder of f on division by G is zero.

6. Symmetric polynomials. Recall that, for $k \leq n$ an elementary symmetric polynomial $\sigma_k \in k[x_1, \ldots, x_n]$ is defined as

$$\sigma_r = \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1} \cdots x_{i_r}$$

   (a) Prove that

$$\prod_{i=1}^{n} (y - x_i) = y^n - \sigma_1 y^{n-1} + \cdots + (-1)^{n-1} \sigma_{n-1} y + (-1)^n \sigma_n$$

   (b) Prove that, if σ_k^r denotes the r-th elementary symmetric polynomial in n variables, then $\sigma_k^r = \sigma_k^{n-1} + x_n \sigma_{k-1}^{n-1}$

   (c) Find a formula (e.g. a generating function) for the dimension of the k-linear space of symmetric polynomials in n variables of total degree d.

7. Write the following symmetric functions as polynomials in elementary symmetric functions:

   (a) $\sum_{i \neq j} x_i^2 x_j, \ \sum_{i \neq j} x_i^2 x_j^2$,

   (b) $\prod_{i \neq j} (x_i - x_j)$ for $n = 3$ (discriminant)

   Hint: Use Gauss algorithm or the fact that homogeneous symmetric polynomials of degree d are linear combinations of monomials in elementary symmetric functions of the appropriate degree.

8. More symmetric polynomials. Let us define $s_r = \sum_{i=1}^{n} x_i^r$. Prove the following identities

   (a) if s_r^n denotes the respective function in n variables then $s_r^n = s_r^{n-1} + x_n^r$

   (b) $s_r - \sigma_1 s_{r-1} + \sigma_2 s_{r-2} - \cdots + (-1)^{r-1} \sigma_{r-1} s_1 + (-1)^r r \sigma_r = 0$, for $1 \leq r \leq n$

   (c) $s_r - \sigma_1 s_{r-1} + \sigma_2 s_{r-2} - \cdots \pm \sigma_n s_{r-n} = 0$, for $r > n$

   Conclude that, if the characteristic of k is zero or bigger than n, then functions s_r generate the ring of invariants $k[x_1, \ldots, x_n]^{S_n}$.