Commutative Algebra Fall 2014/2015

Problem set VII, for November 18th

Integral extensions, integral closures.

A normal ring is a domain integrally closed in its field of fractions. The normalization of a domain is its integral closure in its field of fractions.

1. Find the integral closure of the following domains in the prescribed fields.
   (a) $\mathbb{C}[x,y]/(y^2 - x^3)$ in its field of fractions; hint: take $t = y/x$,
   (b) $\mathbb{C}[x,y]/(y^2 - x^3 - x^2)$ in its field of fractions; hint: take $t = y/x$,
   (c) $\mathbb{Z}$ in $\mathbb{Q}(\sqrt{n})$, where n is not divided by a square of a prime number; hint: the answer depends on whether $4|n$.
2. True or false, prove or disprove the following statements; all rings are domains.
   (a) If $A \subseteq B$ is an integral extension and $\mathfrak{p} \lhd B$ is a prime ideal then $\mathfrak{p}$ is maximal in B if and only if $\mathfrak{p} \cap A$ is maximal in A.
   (b) A subring of a normal ring is normal.
   (c) If a quotient of a normal ring is a domain then it is normal.
   (d) A localisation of a normal ring is normal.
   (e) If $\overline{A}$ is the normalization of A then the extension $A \subseteq \overline{A}$ is finite.
   (f) If $A \subset B$ is a finite extension and A is Noetherian then B is Noetherian.
   (g) If $A \subset B$ is a finite extension and B is Noetherian then A is Noetherian.
3. Let us assume that $A \subseteq B$ is an integral extension. Prove the following statements:
   (a) If $\mathfrak{q}$ is a prime ideal in B then $B/\mathfrak{q}$ is integral over $A/(A \cap \mathfrak{q})$.
   (b) If $\mathfrak{p}$ is a prime ideal in A then there exists a prime ideal $\mathfrak{q}$ in B such that $\mathfrak{p} = \mathfrak{q} \cap A$.

4. Let us denote $\mathbb{N} = \mathbb{Z}_{\geq 0}$. We consider a semigroup $\Gamma \subset \mathbb{N}^n$ and we define the cone $C(\Gamma) = \mathbb{R}_{\geq 0} \cdot \Gamma \subset \mathbb{R}^n$ generated in $\mathbb{R}^n$ by Γ. We define a ring $k[\Gamma] \subset k[x_1, \dots, x_n]$ which, as as a k-vector space, is spanned on monomials $x^\gamma = x_1^{\gamma_1} \cdots x_n^{\gamma_n}$, for $\gamma \in \Gamma$.

   (a) We define $\overline{\Gamma} = C(\Gamma) \cap \mathbb{Z}^n$. Prove $\overline{\Gamma} = \{\gamma \in \mathbb{Z}^n | \; \exists m > 0 : m\gamma \in \Gamma\}$.

   (b) Prove that the extension $k[\Gamma] \subset k[\overline{\Gamma}]$ is integral. Is it finite?

   (c) Prove that $k[\overline{\Gamma}]$ is integrally closed in $k[\mathbb{N}^n] = k[x_1, \dots, x_n]$.

   (d) Prove that the ring $k[\overline{\Gamma}]$ is normal hence it is the normalization of $k[\Gamma]$.

5. Let k denote an algebraically closed field with char $k \neq 2$, we set $A = k[t]$. For any polynomial $f \in A$, such that $f = \prod(t - a_i)^{n_i}$, with $a_i \neq a_j$ for $i \neq j$ and $n_i \geq 1$, we consider the quotient ring $B = k[t, x]/(x^2 - f)$; the class of x in B we denote by $\sqrt{f}$.

   (a) Find the the necessary and sufficient fondition for B to be a domain. (In what follows we assume that B is a domain.)

   (b) Prove that the integral closure of A in the field of fractions (B) is equal to the normalization of B.

   (c) Find the necessary and sufficient condition for B to be normal.