Commutative Algebra Fall 2014/2015

Problem set VI, for November 11th
We celebrate 11/11 with monomial ideals.

Let k be a field of characteristic 0. An ideal $I \lhd k[x_1, \dots, x_n]$ is called monomial if it is generated by monomials in $x_1, \dots, x_n$.

1. Prove: an ideal is monomial if and only if, as a vector space over k, it is spanned on monomials in $x_1, \dots, x_n$.

2. Prove: the minimal set of generators of a monomial ideal is uniquely defined and it is finite.

3. True or false? Prove or disprove: if I and J are monomial ideals then $I + J$, $I \cap J$ and $I \cdot J$ are also monomial ideals.

4. Let I be a monomial ideal. Prove the following:

   (a) I is prime if and only if $I = (x_{i_1}, \dots, , x_{i_r})$,

   (b) I is primary if and only if there exist $x_{i_1}, \dots, x_{i_r}$ such that I contains monomials only in these variables and I contains $x_{i_1}^{d_1}, \dots, , x_{i_r}^{d_r}$ for some $d_i \geq 1$,

   (c) $I = \sqrt{I}$ if and only if I is generated by square-free monomials.

5. Let a monomial m be in the set of minimal generators of a monomial ideal I. Suppose that $m = m_1 m_2$ where m_1 and m_2 have no non-unit common factor. Prove that $I = (I + (m_1)) \cap (I + (m_2))$.

6. Prove that a monomial ideal can be presented as an intersection of primary monomial ideals. Show an algorithm which yields such a presentation.

7. Let I_1 and I_2 be two monomial ideals generated by monomials in disjoint sets of variables, say in $x_1, \dots, x_s$ and $x_{s+1}, \dots, x_r$, respectively. Prove that $I_1 \cap I_2 = I_1 \cdot I_2$. What if we drop the assumption on generating monomials being in disjoint sets of variables?

8. Let Γ be a graph with n vertices denoted by x_i, where $i = 1, \dots, n$. A subset $V \subseteq \{x_1, \dots, x_n\}$ is a a vertex cover of Γ if every edge of Γ is contained in at least one of the vertices from V. We consider an ideal

$I_\Gamma \lhd k[x_1, \ldots, x_n]$ which is generated by $x_i x_j$ such that x_i is adjacent to x_j in Γ. For V as above we define $I_V = (x_i \in V)$. Prove that $I_\Gamma = \bigcap_{V \in \mathcal{C}} I_V$ where $\mathcal{C}$ is the set of all minimal vertex covers of Γ.