Commutative Algebra Fall 2014/2015

Problem set V, for November 4th
Noetherian rings, Artinian rings.

An ideal I is called indecomposable if from its presentation as an intersection $I = J_1 \cap J_2$, where J_i's are ideals, it follows that either $I = J_1$ or $I = J_2$.

1. Let $\mathcal{C}^{\infty}_{\mathbb{R}^n,0}$ be a stalk at 0 of germs of C^{∞} real functions on $\mathbb{R}^n$. Prove that this is a local ring with the maximal ideal of functions vanishing at 0. Show that it is not Noetherian.

2. Let A be a Noetherian ring. Prove that if $f : A \to A$ is a surjective homomorphism then is also injective. Hint: use the sequence of ideals $\ker f \subseteq \ker f^2 \subseteq \cdots$.

3. Let A be a Noetherian ring. Prove that any ideal in A is an intersection of a finite number of indecomposable ideals. Hint: consider a family of ideals which can not be presented as an intersection of a finite number of indecomposable ideals.

4. Let A be a Noetherian ring. Prove the following: if the zero ideal in A is indecomposable then it is primary. Hint: suppose that $ab = 0$ and $b \neq 0$, prove that $(a^n) \cap (b) = (0)$ for some $n > 0$, use the ideals annihilating x^n.

5. Another version of Nakayama lemma. Let A be a local ring with the maximal ideal $\mathfrak{m}$ and the residue field $k = A/\mathfrak{m}$. Take a finitely generated A-module M. Prove that elements $m_1, \ldots, m_r \in M$ generate M over A if and only if their classes in $M/\mathfrak{m}M$ generated the k-vector space $M/\mathfrak{m}M$.

6. Let $(A, \mathfrak{m})$ be a local Noetherian ring. Prove that a finitely generated A module is flat if and only if it is free.

7. A topological space is called Noetherian if any family of closed subsets has a minimal element; show that it is equivalent to the ascending chain condition for open sets. Prove that any Noetherian topological space is quasi-compact. Prove that the spectrum of a Noetherian ring with Zariski topology is a Notherian topological space.

8. Artinian rings. A ring A is Artinian if it satisfies the descending chain condition: every descending chain of ideals stabilizes. Prove the following statements about an Artinian ring A:

   (a) every prime ideal in A is maximal

   (b) the nilradical of A coincides with its Jacobson ideal

   (c) A has only a finite number of maximal ideals

9. Let A be a finitely generated algebra over a field k. Prove that A is Artinian if and only if it is finite over k.

10. Let A be a Noetherian ring. Prove that it is Artinian if and only if its spectrum is finite and discrete.