Commutative Algebra Fall 2014/2015

Problem set IV, for October 28th

Modules: the category of modules over a ring A is denoted by $\mathcal{M}od_A$.

1. Euler characteristic. A function $\chi : \mathrm{Obj}(\mathcal{M}od_A) \to \mathbb{Z}$ is called additive if for any short exact sequence of A-modules $0 \to M_1 \to M_2 \to M_3 \to 0$ it holds $\chi(M_2) = \chi(M_1) + \chi(M_3)$. Prove that for any exact sequence of A-modules $0 \to M_1 \to M_2 \to \cdots \to M_r \to 0$ the alternating sum $\sum_{i=1}^{r} (-1)^i \cdot \chi(M_i)$ is zero.

2. Five lemma. Consider a commutative diagram of A-modules with exact rows:
$$\begin{array}{ccccccccc} M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & M_4 & \longrightarrow & M_5 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ N_1 & \longrightarrow & N_2 & \longrightarrow & N_3 & \longrightarrow & N_4 & \longrightarrow & N_5 \end{array}$$
Show that the central arrow is an isomorphism if the other four are isomorphisms.

3. Snake lemma. Consider the following commutative diagram of A-modules with exact rows
$$\begin{array}{ccccccc} & & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N_1 & \longrightarrow & N_2 & \longrightarrow & N_3 & & \end{array}$$
The vertical arrows are denoted by α_1, α_2 and α_3, respectively. Prove that there exists $\delta : ker(\alpha_3) \to coker(\alpha_1)$ which makes the following natural sequence exact:
$$ker(\alpha_1) \to ker(\alpha_2) \to ker(\alpha_3) \xrightarrow{\delta} coker(\alpha_1) \to coker(\alpha_2) \to coker(\alpha_3)$$

4. Prove that a sequence of A-modules $M_1 \to M_2 \to M_3 \to 0$ is exact if and only if for every A-module N the $\mathrm{Hom}(\,\cdot\,, N)$ induced sequence
$$0 \longrightarrow \mathrm{Hom}(M_3, N) \longrightarrow \mathrm{Hom}(M_2, N) \longrightarrow \mathrm{Hom}(M_1, N)$$
is exact. Prove a similar statement for $0 \to M_1 \to M_2 \to M_3$ and $\mathrm{Hom}(N, \,\cdot\,)$.

5. Consider the exact sequence of $\mathbb{Z}$-modules $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_r \to 0$ where $\mathbb{Z} \to \mathbb{Z}$ is given by multiplication by r. Apply the functor $\mathrm{Hom}(\,\cdot\,, \mathbb{Z})$ to this sequence and conclude that the result can be completed to a similar sequence. What will be the result if you apply $\mathrm{Hom}(\,\cdot\,, \mathbb{Z}_s)$ where s is coprime with r?

6. Calculate the $\mathbb{Z}$-module $\mathbb{Z}_r \otimes_{\mathbb{Z}} \mathbb{Z}_s$ where r and s are coprime.

7. Recall that an A-module M is flat if the tensor multiplication $\otimes_A M$ is an exact functor (or, equivalently, it preserves injectivity). Prove that every finitely generated free module is flat. Prove that for $\mathbb{Z}$-modules the converse its true: if M is finitely generated and flat then it is free.

8. Exterior product. For a given A-module M show the existence and uniqueness of a module $\bigwedge^r M$ together with an r-linear alternating morphism $\psi : M \times \cdots \times M \to \bigwedge^r M$ such that for any r-linear alternating morphism $\phi : M \times \cdots \times M \to N$ there exists exactly one homomorphism $\widetilde{\phi} : \bigwedge^r M \to N$ satisfying the condition $\widetilde{\phi} \circ \psi = \phi$. Notation: for a given r-tuple $m_1, \ldots, m_r \in M$ by $m_1 \wedge \cdots \wedge m_r$ we denote $\psi(m_1, \ldots, m_r)$.

9. Symmetric product. For a given A-module M show the existence and uniqueness of a module $S^r M$ together with an r-linear symmetric morphism $\psi : M \times \cdots \times M \to S^r M$ such that for any r-linear symmetric morphism $\phi : M \times \cdots \times M \to N$ there exists exactly one homomorphism $\widetilde{\phi} : S^r M \to N$ satisfying the condition $\widetilde{\phi} \circ \psi = \phi$.

10. Determinant. Let M be a free A-module of rank d with basis $m_1, \ldots, m_d$. Prove that there exists a unique isomorphisms of A-modules $det : \bigwedge^d M \to A$ such that $det(m_1 \wedge \cdots \wedge m_d) = 1$. Prove that for a arbitrary $r > 0$ the module $\bigwedge^r M$ is free; find its rank.

11. For M as above prove that $S^r M$ is isomorphic with A module of homogeneous polynomials over A of degree r in d variables.

12. Koszul complex. Let M be a finitely generated A-module. we take a non-zero A-homomorphism $h : M \to A$. For $r \geq 0$ we define a homomorphism $d_r : \bigwedge^r M \to \bigwedge^{r-1} M$ as follows (note that $\bigwedge^0 M = A$)

$$d_r(m_1 \wedge \cdots \wedge m_r) = \sum_{i=1}^{r} (-1)^{i+1} h(m_i) \cdot m_1 \wedge \cdots \wedge m_{i-1} \wedge m_{i+1} \wedge \cdots \wedge m_r$$

(a) Show that $d_i \circ d_{i+1} = 0$ so that $im(d_i) \subset ker(d_{i-1})$.

(b) Prove that if $A = k$ is a field then $im(d_i) = ker(d_{i-1})$ for $i > 1$.

(c) Discuss the condition $im(d_i) = ker(d_{i-1})$ for $A = k[x, y]$, $M = A \oplus A$ and $h(f_1, f_2) = xf_1 + yf_2$.

(d) Is always $im(d_i) = ker(d_{i-1})$?