Commutative Algebra Fall 2014/2015

Problem set III, for October 21th
Presheaves and sheaves, direct and inverse limits.

Presheaf. Let (X, τ) be a topological space. A presheaf $\mathcal{S}$ (of sets, abelian groups or vector spaces, etc) over X we call a function $\tau \ni U \to \mathcal{S}(U)$ which for every open set U associates (a set, an abelian group, a vector space, etc) $\mathcal{S}(U)$. By default, we will assume that $\mathcal{S}(\emptyset)$ is the terminal object in the category. Moreover for every pair of open sets $U \subseteq V$ we have a morphism $r_{VU} : \mathcal{S}(V) \to \mathcal{S}(U)$ such that for any triple $U \subseteq V \subseteq W$ it holds $r_{VU} \circ r_{WV} = r_{WU}$. Elements of $\mathcal{S}(U)$ are called sections of $\mathcal{S}$ over U; the morphism r_{VU} is called the restriction.
Sheaf. A presheaf $\mathcal{S}$ is called a sheaf if, in addition, for every open covering $(U_\alpha)_{\alpha\in\Lambda}$ of an open $V = \bigcup_{\alpha\in\Lambda}$ and every family of sections $s_\alpha \in \mathcal{S}(U_\alpha)$ satisfying the following condition

$$\forall_{\alpha,\beta\in\Lambda} \quad r_{U_\alpha,U_\alpha\cap U_\beta}(s_\alpha) = r_{U_\beta,U_\alpha\cap U_\beta}(s_\beta)$$

there exists exactly one section $s \in \mathcal{S}(V)$ such that $r_{VU_\alpha}(s) = s_\alpha$ for every $\alpha \in \Lambda$.
Direct limit and inverse limit. We consider a category $\widehat{S} = (S, \leq)$ of a set S with partial order $\leq$. Take a functor (indexing) $\widehat{S} \to \mathcal{C}$ assigning indexes $S \ni s \to A_s \in \mathrm{Obj}_{\mathcal{C}}$ and morphisms for $s \leq s'$ denoted by $\phi_{s's} \in \mathrm{Mor}_{\mathcal{C}}(A_{s'}, A_s)$. The inverse limit of the system $(A_s)_{s\in S}$ is an object $D \in \mathrm{Obj}_{\mathcal{C}}$ together with morphisms $\psi_s : D \to A_s$ such that for every $\phi_{s's} \in \mathrm{Mor}_{\mathcal{C}}(A_{s'}, A_s)$ we have $\psi_s = \phi_{s's} \circ \psi_{s'}$. Moreover D is a terminal object satisfying this condition in the sense that if D' satisfies the above condition for D then there exists a morphism $D' \to D$ which makes the appropriate compositions commute. The inverse limit is denoted by $\varprojlim A_s$. The direct limits is defined by inverting arrows and denoted by $\varinjlim A_s$.

1. Verify that a presheaf on a topological space (X, τ) is a contravariant functor from category of open sets with inclusions to a category $\mathcal{S}et$, $\mathcal{A}b$ etc.

2. Constant sheaf. Let G be any abelian group and (X, τ) a topological space. For every $U \in \tau$ define $\mathcal{G}(U) := G$ and for $U \subset V$ we set $r_{VU} = id_G$. Show that this defines a pre-sheaf $\mathcal{G}$ over X. Is this a sheaf?

3. Let G and X be as above. For every $U \in \tau$ we define $\widetilde{\mathcal{G}}(U) := G^U = \{f : U \to G\}$. For $V \subset U$ the restriction map $\widetilde{\mathcal{G}}(U) \to \widetilde{\mathcal{G}}(V)$ is just restriction of functions. Does this define a (pre)sheaf?

4. Let (X, τ) be a topological space. For $U \in \tau$ by $\mathcal{C}(U)$ we denote the set of continuous $\mathbb{R}$ valued functions with addition and multiplication defined value-wise. Prove that $\mathcal{C}$ with natural restrictions is a sheaf. Verify a similar statement for the sheaf $\mathcal{C}^\infty$ defined on a differentiable manifold.

5. Show that fiber product and co-product are special cases of direct and inverse limits.

6. Let $(\mathbb{Z}_{>0}, \leq)$ be the set of positive integers with the natural order. The following systems of groups (rings) are indexed by $(\mathbb{Z}_{>0}, \leq)$. Verify whether they admit (direct or inverse) limit

   (a) $\mathbb{Z} \hookrightarrow p^{-1}\mathbb{Z} \hookrightarrow p^{-2}\mathbb{Z} \hookrightarrow p^{-3}\mathbb{Z} \hookrightarrow \cdots$ with p being a prime number,

   (b) $k[x]/(x) \leftarrow k[x]/(x^2) \leftarrow k[x]/(x^3) \leftarrow \cdots$

7. Let A be a domain with a prime divisor $\mathfrak{p}$. On the set $A \setminus \mathfrak{p}$ we set a partial order $b' \leq b \Leftrightarrow b' \mid b$. Consider the localization $A_b = \{a/b^r : a \in A, r \geq 0\} \subset (A)$; note that for $b' \mid b$ we have $A_{b'} \hookrightarrow A_b$. Prove that $A_\mathfrak{p}$, that is the localization with respect to the multiplicative system $A \setminus \mathfrak{p}$, is a direct limit of $(A_b)_{b \in A \setminus \mathfrak{p}}$.

8. A stalk and a germ. Let $\mathcal{S}$ be a presheaf (of abelian groups etc) over (X, τ). For $x \in X$, let τ_x denotes the set of open sets in X containing x with the partial order determined by inclusion. Prove that there exists a direct limit of the system $(\mathcal{S}(U))_{U \in \tau_x}$, we call it the stalk of $\mathcal{S}$ in x and denote $\mathcal{S}_x$. Prove that it inherits the algebraic properties of $\mathcal{S}$. The class of (or the image of) $s \in \mathcal{S}(U)$ in $\mathcal{S}_x$ for $x \in U$ is denoted by s_x and called the germ of s in x. What are the stalks in exercises 2 and 3?

9. In the situation of the previous exercise define the weakest topology on the disjoint union of stalks $\coprod_{x \in X} \mathcal{S}_x$ such that for any $s \in \mathcal{S}(U)$ the set $\{s_x : x \in U\}$ is open. Call the resulting topological space by $\widetilde{\mathcal{S}}$. Define the projection $\pi : \widetilde{\mathcal{S}} \to X$ by setting $\pi(s_x) = x$; show

that it is continuous and locally homeomorphism. Prove that $s \in \mathcal{S}(U)$ determines a continuous $\tilde{s} : U \to \widetilde{\mathcal{S}}$ such that $\pi \circ \tilde{s} = id_U$ (i.e. a section).

10. Sheafification of a presheaf. In the situation of the previous exercise for $U \subset X$ we define

$$\widehat{\mathcal{S}}(U) = \{s : U \to \widetilde{\mathcal{S}} \text{ continuous } \& \ \pi \circ s = id_U\}$$

Prove that $\widehat{\mathcal{S}}$ is a sheaf of groups etc. over X and there exists a natural transformation of contravariant functors $\mathcal{S} \to \widehat{\mathcal{S}}$. What is the sheafification of the presheaf from exercise 2?