Commutative Algebra Fall 2014/2015

Problem set II, for October 14th
Short introduction to categories.

It is assumed that you know basic definitions of this theory. There are many sources for getting them; here are two of them:

- Adamek, Herrlich, Stecker, *Abstract and concrete categories*
- Freyd, *Abelian Categories*

The categories of sets and algebraic structures (groups, abelian groups, commutative rings, vector spaces over a fixed field k), and topological space will be denoted by, respectively, $\mathcal{S}et$, $\mathcal{G}r$, $\mathcal{A}b$, $\mathcal{R}ing$, $\mathcal{V}ect_k$, $\mathcal{T}op$.

1. Define the notion of an isomorphism and of the inverse of a morphism in categorical terms.

2. An object A in a category $\mathcal{C}$ is called initial if it has exactly one morphism into every object of $\mathcal{C}$. Similarly we define the terminal object of $\mathcal{C}$. Prove: if an initial (terminal) object exists then it is unique up to isomorphism. Check whether the categories listed above have the initial (terminal) object.

3. Prove that abelianization is a covariant functor $Ab : \mathcal{G}r \to \mathcal{A}b$. Is this a full functor? Is this a faithful functor?

4. Let $(S, \leq)$ be a set with partial order. Show that $(S, \leq)$ determines a category with objects being elements from S and morphisms determines by the order.

5. Let $\mathcal{C}$ be a category with a distinguished object A. Show that mappings $B \to \mathrm{Mor}_{\mathcal{C}}(A, B)$ and

$$\mathrm{Mor}_{\mathcal{C}}(B, C) \ni f \longrightarrow (\ \mathrm{Mor}_{\mathcal{C}}(A, B) \ni g \to f \circ g \in \mathrm{Mor}_{\mathcal{C}}(A, C)\ )$$

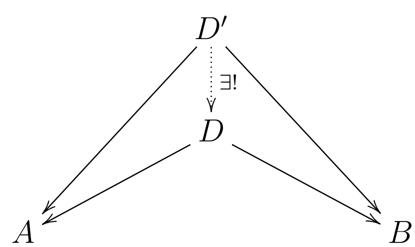
   define covariant functor from $\mathcal{C}$ into $\mathcal{S}et$. We will denote it by $\mathfrak{h}_A : B \to \mathrm{Mor}_{\mathcal{C}}(A, B)$. Similarly we define a covariant functor $\mathfrak{h}^A : B \to \mathrm{Mor}_{\mathcal{C}}(B, A)$. Prove that any morphism $A' \to A$ determines natural transformations of functors $\mathfrak{h}_A \to \mathfrak{h}_{A'}$ and $\mathfrak{h}^{A'} \to \mathfrak{h}^A$.

6. Yoneda lemma. For a given category $\mathcal{C}$ we define its functor category $\widehat{\mathcal{C}}$ whose objects are contravariant functors $\mathcal{C} \to \mathcal{S}et$ and morphisms are natural tranformations of such functors. Let us consider a functor $\widehat{\Phi} : \mathcal{C} \to \widehat{\mathcal{C}}$ such that $\widehat{\Phi}_{\mathrm{Obj}}(A) = \mathfrak{h}^A$ while on morphisms $\widehat{\Phi}$ is as defined in the previous exercise. Show that $\widehat{\Phi}$ is faithful and full.

7. A contravariant functor $\mathcal{C} \to \mathcal{S}et$ is called representable by an object A in $\mathcal{C}$ if it is naturally isomorphic to the functor $\mathfrak{h}^A$. Show that the forgetful functor $\mathcal{A}b \to \mathcal{S}et$ is naturally isomorphic to $\mathfrak{h}_{\mathbb{Z}}$.

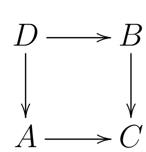
8. Let $vect_k^{<\infty}$ denote category in which objects are linear spaces k^n (one object for every $n \geq 0$) while $\mathrm{Mor}_{vect}(k^n, k^m)$ are matrices $n \times m$ with multiplication as composition. Show that the embedding of $vect_k^{<\infty}$ into $\mathcal{V}ect_k^{<\infty}$ is equivalence of categories.

9. Categorical product of two objects $A,\ B \in \mathrm{Obj}_{\mathcal{C}}$ is an object D with two morphisms $A \leftarrow D \to B$ such that for any object D' with morphisms $A \leftarrow D' \to B$ there exists exactly one morphism from D' to D which makes the following diagram commutative

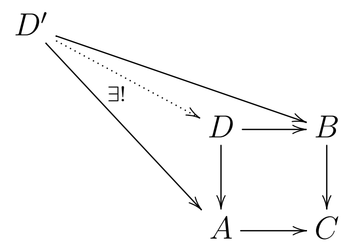


Usually D is denoted by $A \times B$. Co-product is defined similarly by reversing the arrows. Prove that the product (co-product) is unique up to isomorphism. Prove that $A \times B$ is naturally isomorphic to $B \times A$. Describe products and co-products in $\mathcal{S}et$, $\mathcal{T}op$, $\mathcal{A}b$, $\mathcal{G}r$, $\mathcal{R}ing$; note that they do not have to exist.

10. In category $\mathcal{C}$ we consider three objects with two morphisms $A \to C \leftarrow B$. The fiber product of A and B over C is an object D with morphisms $A \leftarrow D \to B$ which make this diagram commutative



Moreover if D' satisfies the above condittion formulated for D then there exists a unique morphism $D' \to D$ which makes the following diagram commutative



We denote such product (if it exists) by $A \times_C B$. The fiber co-product is defined similarly be reversing the arrows. Show the uniqueness (up to isomorphism) of the fiber product and co-product. Contemplate the fiber products and co-products in $\mathcal{S}et$, $\mathcal{T}op$, $\mathcal{A}b$. Prove that if $\mathcal{C}$ has the terminal object Z and admits the product $A \times B$ then the fiber product $A \times_Z B$ is isomorphic to $A \times B$.

11. Composition of fiber products is a fiber product. Suppose that the following arrows are defined

$$
\begin{array}{ccccc}
E \times_A (A \times_C B) & \longrightarrow & A \times_C B & \longrightarrow & B \\
\downarrow & & \downarrow & & \downarrow \\
E & \longrightarrow & A & \longrightarrow & C
\end{array}
$$

Prove that $E \times_A (A \times_C B)$ is isomorphic to $E \times_C B$ where $E \to C$ is the composition $E \to A \to C$.

12. Fiber product base change. Suppose that there exists a fiber product for $A \to C \leftarrow B$. Prove that for every $C \to E$ we have a natural morphism $A \times_C B \to A \times_E B$ and the diagram

$$
\begin{array}{ccc}
A \times_C B & \longrightarrow & A \times_E B \\
\downarrow & & \downarrow \\
C & \longrightarrow & C \times_E C
\end{array}
$$

where $C \to C \times_E C$ is the diagonal which means the morphism coming from two identities $C \to C \leftarrow C$. Prove that the above diagram makes the fiber product.