Commutative Algebra Fall 2014/2015

Problem set I, for October 7th

Brief review on rings and ideals. Most of the problems are standard and can be found (with solutions) in textbooks. You need to recall the appropriate definitions by yourself. Our default source is the Atiyah – Macdonald book; most problems are actually taken from this book. All rings are commutative and with the unit element 1.

1. Prove that if a finite ring is an integral domain then it is a field.
2. Prove that an ideal $\mathfrak{p}$ in a ring R is prime or maximal if and only if $R/\mathfrak{p}$ is an integral domain or a field, respectively.
3. Prove: if $a \in A$ is nilpotent then $1 + a$ is invertible.
4. Consider a polynomial $f = a_0 + a_1 t + a_2 t^2 + \cdots + a_d t^d$ in the ring $A[t]$ of polynomials with coefficients in A. Prove the following: (1) f is invertible in $A[t]$ if and only if a_0 is invertible and a_i's are nilpotent in A for $i = 1, \ldots, d$, (2) f is nilpotent if and only $a_0, \ldots, a_d$ are nilpotent in A, (3) f is a zero divisor if exists $a \in A$ such that $af = 0$.
5. Prove that the set of all nilpotent elements in a ring R is an ideal which is the intersection of all prime ideals in R (we will call it nil-radical).
6. The radical of an ideal $\mathfrak{a} \lhd R$ is $\sqrt{\mathfrak{a}} = \{a \in A : \exists_{n>0} \ \ a^n \in \mathfrak{a}\}$. Prove that the radical of an ideal $\mathfrak{a}$ is the intersection of all prime divisors which contain $\mathfrak{a}$.
7. Prove that for any ideals $\mathfrak{a}$, $\mathfrak{b} \lhd A$ it holds $\sqrt{\sqrt{\mathfrak{a}}} = \sqrt{\mathfrak{a}}$, $\sqrt{\mathfrak{a} + \mathfrak{b}} = \sqrt{\sqrt{\mathfrak{a}} + \sqrt{\mathfrak{b}}}$, $\sqrt{\mathfrak{a} \cap \mathfrak{b}} = \sqrt{\mathfrak{a}\mathfrak{b}} = \sqrt{\mathfrak{a}} \cap \sqrt{\mathfrak{b}}$.
8. An ideal $\mathfrak{q}$ is called primary if from $ab \in \mathfrak{q}$ it follows that $a \in \mathfrak{q}$ or $b^n \in \mathfrak{q}$ for some $n > 0$. Show that the radical of a primary ideal is prime. Show that $\mathfrak{q} \lhd A$ is primary if and only if all zero divisors in $A/\mathfrak{q}$ are nilpotents.
9. For a ring A by $\mathrm{Spec}(A)$ we denote the set of prime ideals in A; we call it the spectrum of A. For any subset $Z \subset A$ we define $V(Z) =$

$\{\mathfrak{p} \in \mathrm{Spec}(A) : \mathfrak{p} \supset Z\}$. Prove that the operation $Z \mapsto V(Z)$ satisfies the following properties: (0) if $\mathfrak{a} = (Z)$ then $V(Z) = V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$, (1) $V(\{0\}) = \mathrm{Spec}(A)$, $V(A) = \emptyset$, (2) for any two ideals $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$, (3) for any family Z_i of subsets of A it holds $V(\bigcup_i Z_i) = \bigcap_i V(Z_i)$. Conclude that on $\mathrm{Spec}(A)$ we have a topology in which the sets of type $V(Z)$ are closed; we call this topology Zariski topology.

10. Situation as above. For each $a \in A$ by U_a we denote the complement of $V(\{a\})$ in $\mathrm{Spec}(A)$. Prove that the sets U_a form a basis of the Zariski topology on $\mathrm{Spec}(A)$.

11. The intersection of all maximal ideals in A is called the Jacobson ideal of A and is denoted by $\mathfrak{J}(A)$. Prove that $a \in \mathfrak{J}(A)$ if and only if for every $b \in A$ the element $1 - ab$ is invertible in A.