Notes for the Logic and Metaphysics course

May 17, 2017

1 Propositional modal logic

1.1 Classical Propositional Logic: a Brief Remainder

Language In these notes V will denote a fixed infinite set, whose all elements can be arranged in an infinite sequence whose positions are given by natural numbers (such sets are called *countable*; there are sets that do not have this property, e.g. the set of real numbers. They are called *uncountable*). Elements of V are called *propositional variables* and are denoted p, p_1, $p_2, \ldots$, q, $q_1, \ldots, r, r_1, \ldots$.

Definition 1 (Formulae of CPL)**.** The set of formulae F_{prop} of CPL is the least set P such that

1. $V \subseteq F_{prop}$
2. if ϕ, ψ are any elements of F_{prop}, then so are $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\neg\phi)$ and $(\phi \to \psi)$.

The clause "F_{prop} is the least set P such that conditions 1. and 2. hold" means that F_{prop} satisfies 1 and 2 and it is a subset of every set which satisfies 1. and 2.

Semantics Now we proceed to the semantics of CPL. Definition is surprisingly simple and we will explain its meaning in a moment.

Definition 2 (Models for CPL)**.** A model M for CPL is a subset of V.

Interpretation: Definition becomes more intuitive if we think of elements of V as atomic facts that can hold independently of each other. A model is then a specification which atomic facts hold (i.e. elements of M) and which do not (i.e. those left outside M).

Let us note that to define a subset of V is *essentially* the same as to define a function f mapping each element of V either to 0 or 1 (the choice of 0 and 1 is purely conventional - any two distinct object will do). Indeed, if a subset M of V is given then function f can be defined as sending the elements of M to 1 and the elements not belonging to M to 0. In the other direction, if f is given, then a subset of V can be defined as the set of all those elements which are sent by f to 1. Sometimes we will make use of this double characterization, treating a model M as a function, which we will call a *valuation*.

Definition 3 (Satisfaction relation)**.** Let M be a model for CPL and ϕ be a formula of CPL. We shall say that $M \models \phi$ iff

1. $\phi = p$ for some $p \in V$ and $p \in M$, or
2. $\phi = \psi_1 \wedge \psi_2$ and $M \models \psi_1$ and $M \models \psi_2$, or
3. $\phi = \psi_1 \vee \psi_2$ and $M \models \psi_1$ or $M \models \psi_2$, or
4. $\phi = \psi_1 \to \psi_2$ and if $M \models \psi_1$, then $M \models \psi_2$, or
5. $\phi = \neg\psi$ and it is not the case that $M \models \psi$.

Example 4. Let $M = \{p, q\}$ then

$$M \models p \wedge ((p \to r) \to q)$$

since:

1. $M \models p$
2. $M \not\models r$
3. $M \not\models p \to r$ (because of the two above observations)
4. $M \models ((p \to r) \to q)$ (because the antecedent is false)

Proof System

Definition 5 (Hilbert-Style Proof System)**.** *Axiom* is any sentence of the form

1. $\neg\phi \to (\phi \to \psi)$
2. $\phi \to (\psi \to \phi)$

3. $(\phi \to \psi) \to ((\neg\phi \to \psi) \to \psi)$

4. $(\phi \to (\psi \to \theta)) \to ((\phi \to \psi) \to (\phi \to \theta))$

more precisely: any formula resulting from substituting elements of F_{Prop} for ϕ, ψ, θ in one of the above is our axiom. For example

$$(\neg(p \to p)) \to ((p \to p) \to (p \wedge q))$$

is an axiom because it results from the substitution $\phi \mapsto (p \to p), \psi \mapsto (p \wedge q)$. We adopt also a single rule of reasoning - Modus Ponens

$$\frac{\phi, \phi \to \psi}{\psi}$$

A proof of ϕ is any sequence

$$\psi_1, \psi_2, \ldots, \psi_n$$

of elements of F_{Prop} such that

1. each of $\psi_1, \ldots, \psi_n$ is either an axiom or can be obtained via Modus Ponens rule from previous elements in our sequence;

2. $\psi_n = \phi$.

Sentence ϕ *is provable* if and only if there exists a proof of ϕ. We shall denote the symbol

$$\vdash \phi$$

to denote that ϕ is provable.

It can be checked that Hilbert-Style Proof System as defined above matches the natural semantics that we gave earlier. More precisely it holds that:

Theorem 6 (Completeness theorem for Hilbert-Style Calculus)**.** *For every* ϕ *from* F_{prop}

$$\vdash \phi \text{ if and only if for every model } M, M \models \phi$$

The left-to-right part of the above theorem states that the proof system we defined is *sound* - whatever can be deduced in it is true in every "possible world" (i.e. every model for our language). The right-to-left direction states that it is *complete*: whatever is true in every possible world can be justified by reasoning formalized in this calculus.

Example 7. The calculus we defined is not very convenient to work in. For example the following sequence of formulae is a proof of $p \to p$:

1. $(p \to ((p \to p) \to p)) \to ((p \to (p \to p)) \to (p \to p))$ (instantiation of axiom 4 with $\phi \mapsto p$, $\psi \mapsto (p \to p)$, $\theta \mapsto p$)
2. $(p \to ((p \to p) \to p))$ (instantiation of axiom 2 with $\phi \mapsto p$, $\psi \mapsto (\to p)$)
3. $((p \to (p \to p)) \to (p \to p))$ (application of Modus Ponens to the two formulae above)
4. $(p \to (p \to p))$ (insantiation of axiom 2 with $\phi \mapsto p$, $\psi \mapsto p$)
5. $(p \to p)$ (application of Modus Ponens to the two above formulae.)

However it is often used because of the very short definition and some additional theoretical features. We will give a more convenient decision procedure (algorithm) for deciding whether a formula is valid.

Let us observe that if we have two models $M \subseteq V, N \subseteq V$ that *agree* on set of variables $\{p_0, \ldots, p_n\}$, i.e. for each i, p_i belongs to M if and only if p_i belongs to N, then for every formula ϕ which uses only $\{p_0, \ldots, p_n\}$ as propositional variables

$$M \models \phi \text{ if and only if } N \models \phi$$

For example if $M = \{p_0, p_1, p_2\}$, $N = \{p_0, p_1, q_0\}$ then these models satisfy the same formulae with propositional variables p_0, p_1, p_3. For example

$$M \models (p_0 \to \neg p_1) \wedge \neg p_3 \text{ if and only if} N \models (p_0 \to \neg p_1) \wedge \neg p_3$$

It follows that to check whether a formula ϕ is satisfied in a model M we need only finitely many informations about M: for each propositional variable which occurs in ϕ (there are finitely many of them!) we have to know whether it belongs to M or not. Hence it is enough to check each of finitely many possible cases. For example let $\phi = (p \wedge q) \to r$. Then it is sufficient to check what happens if

1. all three p, q, r belong to our model.
2. from p, q, r only p belongs to our model. (similarly for q, r)
3. from p, q, r only p, q belong to our model (similarly for (q, r), (p, r))
4. none of p, q, r belongs to our model.

Let us note that this corresponds to considering all possible assignmets of $0, 1$ (0 for "does not hold", 1 for "holds") to $\{p, q, r\}$. This gives rise to the method of "truth tables": to check whether formula is valid (provable in our Proof System) it is enough to check whether every valuation of its propositional variables satisfies it.

Example 8. Let us check that one of axioms of the Hilbert-Style Proof System is valid by checking all valuations. Let us consider

$$\phi = p \to (q \to p)$$

ϕ contains only two propositional variables p, q. There are four valuations that we have to consider

1. $\begin{cases} p \mapsto 1 \\ q \mapsto 1 \end{cases}$
   (this corresponds to the situation in which model contains both p and q)
2. $\begin{cases} p \mapsto 1 \\ q \mapsto 0 \end{cases}$
   (this corresponds to the situation in which our model contains p but not q)
3. $\begin{cases} p \mapsto 0 \\ q \mapsto 1 \end{cases}$
   (this corresponds to the situation in which a model contains q but not p)
4. $\begin{cases} p \mapsto 0 \\ q \mapsto 0 \end{cases}$
   (this corresponds to the situation in which our model contains neither p nor q)

It is easy to verify that each valuation which sends p to 1 makes ϕ true, since it makes true the implication in the succedent of ϕ. Moreover each valuation which sends p to 0 makes ϕ true since it makes the antecedent of ϕ false. Hence every valuation makes ϕ true, hence ϕ is valid.

The following formula

$$\psi = (p \vee q) \to p \wedge q$$

is not valid since the following valuation

$$\begin{cases} p \mapsto 1 \\ q \mapsto 0 \end{cases}$$

makes ψ false.

In the next exercise $p \leftrightarrow q$ abbreviates $(p \to q) \wedge (q \to p)$.

Exercise 1. Check whether the following formulae are valid. In case they are not valid (i.e. there exists a falsifying valuation) check whether there exists a valuation which makes them true.

1. $(p_0 \to p_0) \to ((\neg p_0) \to p_0)$
2. $(p_0 \to \neg p_0) \to q_0$
3. $((p_0 \vee p_1) \vee p_2) \vee (p_3 \vee (p_4 \vee \neg p_4))$
4. $((p_0 \vee p_2) \wedge \neg p_2) \to p_3$
5. $(p \to q) \to ((p \wedge r) \to q)$
6. $(p \leftrightarrow q) \vee ((q \leftrightarrow r) \vee (p \leftrightarrow r))$
7. $(p \leftrightarrow q) \vee (q \leftrightarrow r)$.
8. $((p \wedge q) \to r) \to ((p \to r) \wedge (q \to r))$
9. $((p \vee q) \to r) \to ((p \to r) \vee (q \to r))$
10. $((p \wedge q) \to r) \to ((p \to r) \vee (q \to r))$

1.2 The syntax of the propositional modal logic

Propositional modal logic is an extension of the classical propositional logic in which to the connectives $\neg, \wedge, \vee, \to$ add two unary (i.e. syntactically behaving like a negation) operators $\Box, \Diamond$ with the intended reading "it is necessary that…" and "it is possible that…" respectively.

Formally, the set of propositional modal formulae over the set of propositional variables V is defined as the smallest set F with the following properties:

1. Any element of V belongs to F.
2. If $\phi \in F$, then $\neg\phi \in F$.
3. If both $\phi \in F$ and $\psi \in F$, then $\phi \wedge \psi \in F$, $\phi \vee \psi \in F$ and $\phi \to \psi \in F$.
4. If $\phi \in F$, then $\Box\phi \in F$, $\Diamond\phi \in F$.

Slightly unwinding the formal definition, we see that propositional modal formulae look like the usual propositional formulae, but we allow to write additional symbols $\Box, \Diamond$ in front of formulae, like negations. We will write the elements of V with the letters $p, q, r, s, \ldots$ or sometimes with subscripts $p_1, p_2, \ldots, q_1, q_2, \ldots$. So as an example of the modal formula we have:

$$\Box(\Diamond p \to \neg\Box\neg q)$$

or

$$\Box(q \vee \Box\Diamond\Box r) \vee (p_1 \to \Diamond\Box\Box p_3).$$

1.3 Kripke models

We will now describe the semantics of the propositional modal logic.

Definition 9. By a **Kripke model** we mean a tuple $\langle K, R, f\rangle$, where

1. K is an arbitrary nonempty set.
2. $R \subset K^2$ is an arbitrary relation.
3. f is an arbitrary function with the domain V, which assigns to every letter $p \in V$ a subset of K.

The above definition is probably somewhat unenlightening, so let's try to elaborate on it. Typically, when we define something as a tuple, we think of such an object as a set with some additional structures: relations, functions etc. defined on this set. In our case, we usually call the elements of K **the possible worlds**, R is called the **accessibility relation** and f is called the **valuation function**.

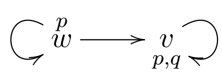
We should think of Kripke models as of sets of worlds for which the relation between w and v holds if and only if v is possible from the point of view of w. For each of these worlds w, we explicitly indicate which atomic facts hold in these worlds — this is what the valuation function does. If a world w belongs to $f(p)$ this intuitively means that the atomic fact p holds in the world w.

We capture this intuition, when relating Kripke models to propositional modal formulae.

Definition 10. Let $\mathcal{K} = (K, R, f)$ be an a arbitrary Kripke model and w a world in this model. We define what does it mean for a modal formula ϕ to be satisfied in the model $\mathcal{K}$ in the world w (which we write $\mathcal{K}, w \models \phi$) by induction on complexity of ϕ.

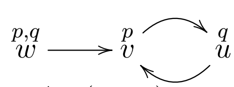
1. If p is a propositional variable, then $\mathcal{K}, w \models p$ iff $w \in f(p)$.
2. $\mathcal{K}, w \models \neg\phi$ iff it is not the case that $\mathcal{K}, w \models \phi$.
3. $\mathcal{K}, w \models \phi \wedge \psi$ iff $\mathcal{K}, w \models \phi$ and $\mathcal{K}, w \models \psi$.
4. $\mathcal{K}, w \models \phi \vee \psi$ iff $\mathcal{K}, w \models \phi$ or $\mathcal{K}, w \models \psi$.
5. $\mathcal{K}, w \models \phi \to \psi$ iff it is not the case that $\mathcal{K}, w \models \phi$ or $\mathcal{K}, w \models \psi$.
6. $\mathcal{K}, w \models \Box\phi$ iff $\mathcal{K}, v \models \phi$ for all v such that $R(w, v)$.
7. $\mathcal{K}, w \models \Diamond\phi$ iff $\mathcal{K}, v \models \phi$ for some v such that $R(w, v)$.

Thus for example in a model $\mathcal{K} = (W, R, f)$ with two worlds w, v such that the relation R holds only between pairs $\langle w, w\rangle$, $\langle w, v\rangle$, $\langle v, v\rangle$ and the valuation function f which assigns $\{w, v\}$ to p and $\{v\}$ to q we have for example: $\mathcal{K}, w \models \neg\Box q \wedge \Box p \wedge \Box\Diamond q$. This model can be depicted

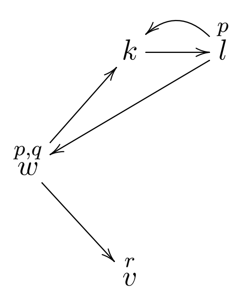


Let us verify that $\mathcal{K}, w \models \Box\Diamond q$, the rest of cases being rather easy. Unfolding the definition we get that $\mathcal{K}, w \models \Box\Diamond q$ if and only if for every world w' which is in relation R with w there exists a possible world w'' in relation R with w' such that w'' satisfy q. This is true in the model given above since only w and v are in the relation with w and each of them sees world v which satisfies q.

Exercise 2. Check whether given formulae hold in the given model at world w:

1. 

   formulae: $\Box p$, $\Diamond\Diamond p$, $\Diamond\Diamond\Diamond p$, $\Diamond\Box(p \vee q)$.

2. 

(w, k, l, v are worlds and p, q, r are propositional variables). Formulae:

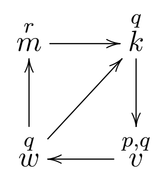
(a) $\Box\Diamond p$,

(b) $\Diamond\Box p$,

(c) $q \to (\Diamond\neg p \to \Diamond\Diamond p)$,

(d) $\Diamond r \to \Box\neg q$

3.



Formulae:

(a) $\Box r$

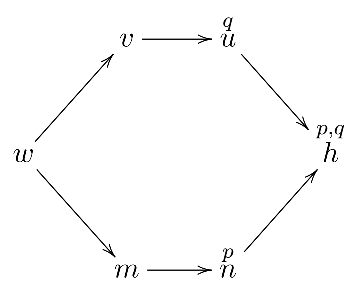
(b) $\Box\Diamond q$

(c) $\Diamond q \to \Box r$

(d) $\Diamond q \to \Box\Diamond q$

Exercise 3. Check whether given formulae hold in the given model at world w:

1.

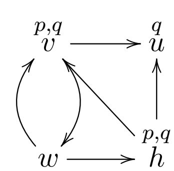


formulae:

(a) $\Diamond\Box p \to \Box\Diamond p$

(b) $\Diamond\Diamond p \to \Diamond\Diamond q$

2.



(a) $\Diamond\Diamond\Diamond\Diamond q \to \Diamond\Diamond\Diamond q$

(b) $\Diamond\Diamond\Diamond\Box q \to \Box\Diamond\Diamond q$

Exercise 4. Give an example of Kripke structures M_i and worlds w_i such that:

1. $M_1, w_1 \models \Box(p \vee q) \wedge \neg(\Box p \vee \Box q)$.
2. $M_2, w_2 \not\models p \to \Diamond p$.
3. $M_3, w_3 \models \Box p \wedge \neg\Box\Box p$.
4. $M_4, w_4 \models \Diamond p \wedge \neg\Box\Diamond p$.
5. $M_5, w_5 \models \Box(p \to q) \to (\Box p \to \Box q)$
6. $M_6, w_6 \models \Diamond\Diamond\Diamond\Diamond p \wedge (\neg\Diamond p \wedge (\neg\Diamond\Diamond p \wedge \neg\Diamond\Diamond\Diamond p))$
7. $M_7, w_7 \Vdash \Diamond\Diamond\Diamond\Box(p \wedge \neg p)$
8. $M_8, w_8 \not\models \Box\Box p \to \Box p$

Exercise 5. Check whether the following formulae are tautologies of Modal Logic, i.e. whether they hold in all Kripke models.

1. $\Box\Diamond p \to \Diamond\Diamond p$
2. $(\Box\Diamond p \wedge \Diamond q) \to \Diamond\Diamond p$
3. $(\Diamond\Box p \wedge \Diamond\Diamond q) \to \Diamond\Diamond p$
4. $\Box(p \wedge q) \to \Box p$
5. $\Box(p \wedge q) \to (\Box p \wedge \Box q)$
6. $\Diamond(p \vee q) \to \Diamond p \vee \Diamond q$
7. $(\Diamond p \wedge \Diamond q) \to \Diamond(p \wedge q)$
8. $(\Box p \equiv \Box q) \to (\Diamond p \equiv \Diamond q)$
9. $\Diamond p \to \Box\Diamond p$
10. $\Box(\Box p \to p) \to \Box p$
11. $\Box(\Diamond p \vee \Diamond\neg p)$.
12. $\Box\big((\Diamond p \wedge \Box(p \to q)) \to \Diamond q\big)$
13. $\Diamond\big((\Diamond p \wedge \Box(p \to q)) \to \Diamond q\big)$
14. $\Box\big((p \wedge (p \to q)) \to \Diamond q\big)$

1.4 Proof system and the most important theories

The next definition introduces a Hilbert-style proof system for modal logic. It is a proper extension of the proof system defined for Classical Propositional Calculus.

Definition 11 (Hilbert-style proof system for ML)**.** Hilbert-style proof system for ML contains as axioms

1. all instantiations of tautologies of Classical Propositional Calculus with formulae of modal logic.
2. all instantiations of the following scheme

$$\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$$

   with formulae of modal logic.

We have two rules of reasoning:

1. Modus Ponens (the same as for Classical Propositional Calculus)
2. Gödels Rule, or Necessitation

$$\frac{\phi}{\Box\phi}$$

Definition of a proof and provability is the same as previously.

Example 12. If ϕ is an instantiation of a tautology of Classical Propositional Calculus, then $\Box\phi$ (and $\Box\Box\phi$, $\Box\Box\Box\phi$...) is provable. For example a proof of $\Box(p \to p)$ is the following sequence of length two

$$p \to p, \Box(p \to p).$$

The above proof system "matches" the semantics for modal logic we introduced earlier - more precisely we have the following theorem. Before stating it it will be convenient to introduce one more definition:

Definition 13. Let $\mathcal{K} = \langle K, R, f\rangle$ be a Kripke model and ϕ a formla of modal logic. We shall say that ϕ is true in $\mathcal{K}$ and write

$$\mathcal{K} \models \phi$$

if for every world $w \in K$ we have

$$\mathcal{K}, w \models \phi$$

Theorem 14 (Completeness Theorem for modal logic)**.** *For every formula ϕ of modal logic*

$$\phi \text{ is provable if and only if for every Kripke model } \mathcal{K}, \mathcal{K} \models \phi$$

This theorem can be strengthened. Let us introduce one natural generalization of the standard provability relation.

Definition 15. Let Th be a set of modal formulae and ϕ a modal formula. We shall say that ϕ *is a consequence of* Th (or that Th proves ϕ) and write Th $\vdash \phi$ if and only if there is a proof of ϕ in which formulae from Th can occur as axioms.

We say that a Kripke model $\mathcal{K}$ satisfies a set of formulae Th (and write it $\mathcal{K} \models$ Th) if it satisfies every formula from Th.

Theorem 16 (Completeness Theorem for modal logic, 2)**.** *For every formula ϕ of modal logic and every set of modal formulae* Th*,*

$$\mathrm{Th} \vdash \phi \text{ if and only if for every Kripke model } \mathcal{K}, \text{if } \mathcal{K} \models \mathrm{Th}, \text{ then } \mathcal{K} \models \phi$$

Moreover we have the following very useful theorem (analogous to the one known from the First Order Logic)

Theorem 17 (Deduction Theorem)**.** *Let ϕ, ψ be modal formulae and* Th *a set of modal formulae. We have that*

$$\mathrm{Th} \vdash \phi \to \psi \text{ if and only if } \mathrm{Th} \cup \{\phi\} \vdash \psi$$

Modal logic introduced so far is a very general tool and can be adapted to model different notions. Proof calculus gives us an easy way to adapt the current formalism to new situations - we might simply add new axioms and investigate into so formed systems. Let us give some examples of systems of modal logic which can be found in the literature:

Definition 18.
1. System K contains all the sentences provable in the Hilbert-style proof calculus for ML.
2. System T contains all the sentences provable in the Hilbert-style proof calculus for ML in which all instantiations of the scheme

$$\Box\phi \to \phi \tag{T}$$

can be taken as axioms.

3. System $S4$ contains all the sentences provable in the Hilbert-style proof calculus for ML in which all instantiations of the schemes

$$\Box\phi \to \phi \tag{T}$$

$$\Box\phi \to \Box\Box\phi \tag{4}$$

can be taken as axioms.

4. System $S5$ contains all the sentences provable in the Hilbert-style proof calculus for ML in which all instantiations of the schemes

$$\Box\phi \to \phi \tag{T}$$

$$\Box\phi \to \Box\Box\phi \tag{4}$$

$$\Diamond\phi \to \Box\Diamond\phi \tag{5}$$

can be taken as axioms.

5. System D contains all the sentences provable in the Hilbert-style proof calculus for ML in which all instantiations of the scheme

$$\Box\phi \to \Diamond\phi \tag{D}$$

can be taken as axioms.

Example 19. Every sentence of the form

$$\phi \to \Box\Diamond\phi \tag{B}$$

is in $S5$ (i.e. it is provable from axiom schemata (T),(4) and (5)). Indeed, let us fix any ϕ. We shall show that $S5 \cup \{\phi\} \vdash \Box\Diamond\phi$ which clearly suffices by Deduction Theorem (Theorem 17). We have $\phi \to \Diamond\phi$ by the contraposition of (T) axiom for $\neg\phi$. Hence by Modus Ponens we have $\Diamond\phi$. Using the instantiation of (5) axiom schema for ϕ we get that $\Diamond\phi \to \Box\Diamond\phi$. Hence by Modus Ponens again we get $\Box\Diamond\phi$, as wanted.

Exercise 6. Show that for arbitrary Kripke model $\mathcal{K} = \langle K, R, f\rangle$ such that R is symmetric[1] and arbitrary formula $\psi = \phi \to \Box\Diamond\phi$ we have

$$\mathcal{K} \models \psi$$

Exercise 7. Show that for arbitrary Kripke model $\mathcal{K} = \langle K, R, f\rangle$ such that R satisfies: for every $w \in K$ there exists w' such that wRw' and arbitrary formula $\psi = \Box\phi \to \Diamond\phi$ we have

$$\mathcal{K} \models \psi$$

[1] i.e. if a world w sees a world w' then w' sees w too. Formally: for every w, w', if wRw', then $w'Rw$.

1.5 Kripke Frames and connections with basic systems of Modal Logic

In the previous section we have seen that for some classes of formulae it is possible to determine whether they hold in a given Kripke model basing on appropriate properties of the accessibility relation only. In this paragraph we shall demonstrate a converse to this phenomenon: we will prove that if certain formulae holds in a Kripke model *regardless of the chosen valuation*, then the accessibility relation has appropriate property. The following definition is a translation of the sentence "formula ϕ holds in a Kripke model regardless of the chosen valuation" to a formal language.

Definition 20. A *Kripke Frame* is a nonempty set K (the universe) and a relation $R \subseteq K^2$. In other words: it is a Kripke model with no valuation function. A formula ϕ is *satisfied* in a Kripke frame (K, R) if and only if for every valuation $f : V \to \mathcal{P}(K)$ (recall that V is a set of propositional variables) we have

$$(K, R, f) \models \phi$$

Abusing the notation a little bit we will use symbol $\models$ for satisfiability in a frame.

Example 21. Let $K = \{a, b\}$ and $R = \{\langle a, a\rangle, \langle a, b\rangle, \langle b, a\rangle\langle b, b\rangle\}$ (hence $R = K^2$; such a graph is also called a clique). Then the formula $\Box p \to \Box\Box p$ is satisfied in (K, R), because we have already showed that it is satisfied in every Kripke model in which the relation is transitive. The formula $\Diamond p$ is not satisfied in (K, R) because for a valuation $f(q) = \emptyset$ for every propositional letter we have

$$(K, R, f), a \not\models \Diamond\phi$$

(to give a counterexample for a satisfiability on a frame it is sufficient to find one valuation and one world such that the resulting Kripke model falsify the formula at a chosen world).

Example 22. The formula $\Box p$ is satisfied in a Kripke Frame $K = \{a, b, c, d\}$ and $R = \emptyset$ because for every valuation f, (K, R, f) is a Kripke model in which every world satisfies $\Box\phi$ for every formula ϕ.

Let K be as previously and $R = \{\langle a, b\rangle, \langle a, a\rangle, \langle c, d\rangle\}$. Then the formula $\Box p \to p$ is not satisfied in (K, R) since for f such that

$$f(p) = \{d\}$$

and $f(p) = \emptyset$ for the rest of propositional variables we have

$$(K, R, f), c \not\models \Box p \to p$$

Generalising the (second part of the) above example we might show the following

Proposition 23. *A frame (K, R) satisfies $\Box p \to p$ if and only if R is reflexive, i.e. for every $k \in K$ it holds that kRk.*

Proof. To be added. □

It turns out that we can find similar properties of the accessibility relation for the rest of additional modal properties we introduced in Definition 18. We will summarize them in the following proposition. By saying that the principle X holds in F we mean that every instantiation of the respective scheme holds at F.

Proposition 24. *Let $F = (V, R)$ be a Kripke frame*

1. *The principle* (T) *holds in F iff R is reflexive, i.e. for any $w \in W$ we have $R(w, w)$.*
2. *The principle* (4) *holds in F iff R is transitive, i.e. for any $w_1, w_2, w_3 \in W$ if $R(w_1, w_2)$ and $R(w_2, w_3)$, then $R(w_1, w_3)$.*
3. *The principle* (B) *holds in F iff R is symmetric, i.e. for any $w, w' \in W$ if $R(w, w')$, then $R(w', w)$ as well.*
4. *The principle* (5) *holds in F iff R is Euclidean, i.e. for any w_1, w_2, w_3 if $R(w_1, w_2)$ and $R(w_1, w_3)$, then $R(w_2, w_3)$.*
5. *The principles* (T),(4),(5) *hold jointly in F iff R is an equivalence relation, i.e. it is symmetric, reflexive and transitive.*

Exercise 8. Prove the above proposition.

Putting the above proposition with the second version of Completeness Theorem (Theorem 16) we get the following Completeness Theorem for the modal systems introduced above.

Theorem 25. *Let ϕ be a modal formula.*

1. *T proves ϕ if and only if for every Kripke model $\mathcal{K} = (K, R, f)$ such that R is reflexive we have $\mathcal{K} \models \phi$.*
2. *B proves ϕ if and only if for every Kripke model $\mathcal{K} = (K, R, f)$ such that R is symmetric we have $\mathcal{K} \models \phi$.*

3. *$S4$ proves ϕ if and only if for every Kripke model $\mathcal{K} = (K, R, f)$ such that R is reflexive and transitive we have $\mathcal{K} \models \phi$.*

4. *$S5$ proves ϕ if and only if for every Kripke model $\mathcal{K} = (K, R, f)$ such that R is an equivalence relation we have $\mathcal{K} \models \phi$.*

1.6 Tableaux for propositional modal logic

In the previous subsection we have defined semantics for the propositional modal logic. We can also introduce this logic via a proof system, i.e. by syntactically defining what constitutes a correct proof of a formula. In our case, the proof system to be defined will have one more feature: for any provable formula we will be able to algorithmically *find* its proof, not only *check* whether something is a proof. Let us first introduce a few (very standard) preparatory notions:

Definition 26. By a (directed) **graph** we mean a pair (V, E), where V is any nonempty set and E is any binary relation on V. A **tree** is a graph in which there is no element v such that for some two elements v_1, v_2 both $E(v_1, v)$ and $E(v_2, v)$ holds and there is exactly one element v such that for no v' $E(v, v')$. A **chain** is a subset $\{v_1, \ldots, v_n\} \subseteq V$ such that $E(v_1, v_2)$, $E(v_2, v_3)$, $E(v_3, v_4), \ldots, E(v_{n-1}, v_n)$. A chain in a tree is called **a branch** if no element can be added to it so that it remains a chain (so it is a maximal chain). By a **labelled** graph with labels from the set A we mean a pair (V, E, f), where (V, E) is a graph and f is a function whose domain is V and whose values are elements of A.

Let us elaborate a bit on the above definitions. When talking about pairs (V, E) we really simply think of a set V with some additional structure: in our case E means that we join some elements of V with arrows. So: we have a set with a couple of arrows from one element to the other. We often call elements of graphs **vertices** and pairs in E **edges**. When graph is labelled we simply put another layer of structure on a graph: on some vertices we write some notes containing extra information.

Intuitively a tree is simply a graph, where from one vertex more than one arrow can go, but no two arrows meet together again in one element. On top of that we require, that there is only one vertex which is not pointed at by any arrow. We call it the **root** of the tree.

Let us define first **de Morgan** form ϕ^M of a propositional modal formula ϕ by induction on the shape of ϕ:

1. $p^M = p$ and $(\neg p)^M = \neg p$ for all propositional variables p.

2. $(\phi \wedge \psi)^M = \phi^M \wedge \psi^M$.

3. $(\phi \vee \psi)^M = \phi^M \vee \psi^M$.

4. $(\Box\phi)^M = \Box\phi^M$.

5. $(\Diamond\phi)^M = \Diamond\phi^M$.

6. $(\phi \to \psi)^M = (\neg\phi)^M \vee \psi^M$.

7. $(\neg(\phi \wedge \psi))^M = (\neg\phi)^M \vee (\neg\psi)^M$.

8. $(\neg(\phi \vee \psi))^M = (\neg\phi)^M \wedge (\neg\psi)^M$.

9. $(\neg\neg\phi)^M = \phi^M$.

10. $(\neg\Box\phi)^M = \Diamond(\neg\phi)^M$.

11. $(\neg\Diamond\phi)^M = \Box(\neg\phi)^M$.

The definition may seem quite awkward, but in fact it is extremely natural: we simply rewrite a give formula so that all negations occur only in front of the propositional variables using de Morgan laws. We also eliminate all occurring implication symbols.

If $\phi = \phi^M$, then we say that ϕ is already in de Morgan form.

Now, instead of giving a proper definition of what we mean by tableaux for the propositional modal logic we will present a procedure of creating tableaux in a somewhat informal way. We have written above that tableaux are used to prove some propositional modal formulae. Actually we use this method to *refute* them. As a by-product of doing so, we construct a model in which the formula in question fails to hold.

So suppose we are given a formula ϕ in de Morgan form we would like to refute. We draw a **box** around the formula (some more formulae will be written inside the same box, so it is good to actually draw the box gradually).

At each point of the construction we will have a tree labelled with formulae, some of which will be grouped in boxes. Now, at every step we may enlarge the tree in the following way:

1. Whenever you see both p and $\neg p$ for some propositional variable p on the same branch and in the same world, you may write $\perp$ at the bottom of that branch in the same box as the last element of the branch. We say that this branch is **closed**.

2. Whenever at some vertex you see a formula $\phi \wedge \psi$ write ϕ and ψ subsequently at the end of each branch in the same box in which $\phi \wedge \psi$ is placed (i.e. at each vertex v in the same box as the node labelled with $\phi \wedge \psi$ that points to no element in the same box you draw two new nodes v' and v'' with arrows from v to v' and from v' to v''. If additionally v pointed at some nodes $v_1, \ldots v_n$ in another boxes you erase these old arrows going from v and draw new arrows from v'' to $v_1, \ldots v_n$). Cross out the instantiation of the formula $\phi \wedge \psi$ you used.

3. Whenever you see a node labelled with a formula $\phi \vee \psi$ you may draw at the same world at the end of each branch passing through the node with this occurrence of $\phi \vee \psi$ two new vertices v', v'' labelled with ϕ and ψ respectively and draw new arrows going from the last node v of the respective branch to v' and v'' (so that v has now two new children). If there were some arrows going from v to nodes $v_1, \ldots, v_n$ in other boxes, you erase these arrows draw a duplicate of each labelled tree starting at v_i and you draw arrows from v' to $v_1, \ldots, v_n$ and from v'' to the respective vertices in the duplicate (in practice you may simply draw arrows both from v' and v'' to $v_1, \ldots, v_n$). Cross out the instantiation of the formula $\phi \vee \psi$ you used.

4. Whenever you see a vertex v labelled $\Box\phi$, then for any node v' which lies in the same branch, but in another box joined by an arrow with the box in which the vertex v was located and which points at no vertex within the same box, you may draw a node v'' pointed at by v' and labelled with ϕ. If additionally v' was pointing at some vertices $v_1, \ldots, v_n$ in other boxes, then you may erase these arrows and draw the new arrows from v'' to $v_1, \ldots, v_n$.

5. Whenever you see a vertex v labelled with $\Diamond\phi$, then for any vertex v' in the same branch which points to no other vertex in the same branch you may draw a new box and a new vertex v'' pointed at by an arrow going from v' and labelled with ϕ. This is the only way in which new boxes are introduced.

You call a formula ϕ **Tableaux-refutable** iff there exists a tree formed according to the above rules, whose root is labelled with $(\neg\phi)^M$ and in which there is a branch which does not close. This means that there exists a Kripke structure in which the formula does not hold.

Now the following fact holds:

Fact 27. *A formula is provable iff it is not Tableaux-refutable.*

Exercise 9. Find Tableaux for the following formulae:

1. $\Box(p \vee q) \to (\Box p \vee \Box q)$
2. $p \to \Box\Diamond p$
3. $\Box(\Box p \to p) \to \Box p$
4. $\Box\Diamond p \to \Diamond\Box p$
5. $\Box(p \to \Diamond(p \wedge \Box p)) \to (\Box p \to \Diamond p)$
6. $\Box\Diamond(p \wedge q)$
7. $\Diamond(p \wedge \neg p) \to \Box q$
8. $\Diamond\Diamond p \to \Diamond p$
9. $\big(\Box\Box p \wedge \Diamond p\big) \to \Box p$
10. $\Diamond(p \wedge \Box(p \to q)) \to (\Box p \to \Box\Box q)$

2 First-Order Modal Logic

2.1 First-Order Logic

Syntax A (relational) *signature* (with constants) σ is a pair of nonempty sets $\mathcal{R}, \mathcal{C}$ called the sets of *relations* and *constants* respectively and a function $\rho : \mathcal{R} \to \mathbb{N}$ which assigns arities to symbols of $\mathcal{R}$. Intuitively ρ says how many objects can stand in the relation from $\mathcal{R}$.

We define relational formulae of the first-order logic by induction on complexity of formulae. Namely, we assume that we have fixed some set of first-order variables V which we will denote x, y, z and occasionally also $x_1, x_2, \ldots, y_1, y_2, \ldots, z_1, z_2, \ldots$. Then we define relational formulae of the fist-order logic over the language $\mathcal{L}$ as the smallest set F satisfying the following conditions:

1. $R(v_1, \ldots, v_n) \in F$, where v_i are either variables from V or constants from $\mathcal{C}$ and R is some relation symbol such that $\rho(R) = n$
2. $v_1 = v_2$ where v_1, v_2 are either variables from V or constants from $\mathcal{C}$.
3. $\phi \wedge \psi \in F$, when $\phi \in F$ and $\psi \in F$.
4. $\phi \vee \psi \in F$, when $\phi \in F$ and $\psi \in F$.

5. $\neg\phi \in F$, when $\phi \in F$.
6. $\forall v \ \ \phi$, when $\phi \in F, v \in V$.
7. $\exists v \ \ \phi$, when $\phi \in F, v \in V$.

Example 28. Suppose $\mathcal{R} = \{P, R\}$, $\mathcal{C} = \{c\}$, $\rho(P) = 1$, $\rho(R) = 2$. Then

$$R(x_1, c)$$

and

$$P(x_2)$$

are well built formulae. If so, then so are

$$R(x_1, c) \wedge P(x_2)$$

and

$$\exists x_1 \forall x_2 \big((R(x_1, c) \wedge P(x_2)\big).$$

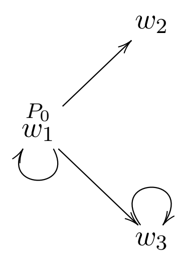
Semantics Let us recall how semantics for first-order logic is defined. Then we will try to use these ideas to define semantics for some first-order version of the modal logic.

Definition 29. Let $\sigma = (\mathcal{R}, \mathcal{C}, \rho)$ be a signature. By a relational model $\mathcal{M}$ we mean any tuple (M, τ), where M is a nonempty set called the **domain** or the **universe** of M and τ is a function with domain $\mathcal{R} \cup \mathcal{C}$ such that

1. for every $R \in \mathcal{R}$, if $\rho(R) = n$, then $\tau(R) \subseteq M^n$
2. for every $c \in \mathcal{C}$, $\tau(c) \in M$.

A model can be thought of as a description of how many basic individuals exist (which objects form the universe) and how are the basic relations from our signature to be interpreted.

Example 30. Let $\sigma = (\mathcal{R}, \mathcal{C}, \rho)$ be such that $\mathcal{R} = \{R\} \cup \{P_i \mid i \in \mathbb{N}\}$, $\rho(R) = 2$, $\rho(P_i) = 1$ for every i and $\mathcal{C} = \emptyset$. Let $M = \{w_1, w_2, w_3\}$ and $\tau(R) = \{(w_1, w_2), (w_1, w_1), (w_3, w_3), (w_1, w_3)\}$ and $\tau(P_0) = a$, $\tau(P_i) = \emptyset$ for $i > 0$. Then M can be depicted as a Kripke model



with P_0 thought of as a propositional variable.

Let us define something more complex: working over the same signature put:

$$
\begin{aligned}
M &= \mathbb{N} \\
\tau(R) &= \{(m,n) \mid m < n\} \\
\tau(P_i) &= \{n \in \mathbb{N} \mid i \text{ divides } n\}
\end{aligned}
$$

Try drawing the above model.

When reading the following definition it is good to keep in mind that what we try to capture is our informal notion of a sentence true of some given structure. This is to be understood in a very straightforward manner in which $\forall x \exists y \; x < y$ is true in the set of natural numbers with its natural order but $\forall x \exists y \; y < x$ is not (why?).

Definition 31. By a valuation α on a first-order model (M, τ) we mean any function whose domain is the set of all first-order variables V and which takes values in the domain of M. It will be convenient to assume that α is defined also on constants from our signature and that for every $c \in \mathcal{C}$ we have $\alpha(c) = \tau(c)$. We define what does it mean for a formula ϕ to hold in a model M under valuation α, in symbols $M, \alpha \models \phi$, by induction on the complexity of formulae:

1. $M, \alpha \models R(v_1, \ldots, v_n)$ iff $(\alpha(v_1), \ldots, \alpha(v_n)) \in \tau(R)$ or, in other words, the elements assigned to $v_1, \ldots, v_n$ by α satisfy the relation R as interpreted in the model M.

2. $M, \alpha \models v_1 = v_2$ iff $\alpha(v_1) = \alpha(v_2)$.

3. $M, \alpha \models (\phi \wedge \psi)$ iff $M, \alpha \models \phi$ and $M, \alpha \models \psi$.

4. $M, \alpha \models (\phi \vee \psi)$ iff $M, \alpha \models \phi$ or $M, \alpha \models \psi$.

5. $M, \alpha \models \neg\phi$ iff it is not the case that $M, \alpha \models \phi$.

6. $M, \alpha \models \exists v\, \phi$ iff there exists a valuation α' which may differ from α at most in what it ascribes to the variable v such that $M, \alpha' \models \phi$.

7. $M, \alpha \models \forall v\, \phi$ iff $M, \alpha' \models \phi$ for all valuations α' which may differ from α at most in what it ascribes to the variable v.

The above inductive definition may seem scary, but it really captures the most intuitive notion of a sentence being true in some structures. We consider the more general case: namely formulae like $R(x, y) \wedge \exists z\, P(z)$. In order to say, whether this formula is satisfied or not in a given model, we have to specify what do we mean by x and y — and this is precisely what the valuations do: they are specification of what variables "mean". It might be confusing that we require valuations to be defined for *all* variables, not only the ones that actually occur in the formula, but this is a purely technical detail to simplify our considerations.

Additionally, we say that a formulae is **true** in a given model iff it is satisfied under all valuations. We denote it $M \models \phi$ (i.e. similarly to the above definition, but omitting the reference to a valuation). It is **valid** iff it is true in all models. Observe that if $\phi(x)$ is a formula in which x is not within the scope of any quantifier (we say that x is a *free variable* in ϕ) then it is true in a model M if and only if $\forall x \phi(x)$ is true in a model M.

Sometimes we will use the following convention: if ϕ contains exactly one free variable, then we instead of writing a model and a valuation on the lefthandside of the satisfiability relation we will be writing simply a model and an *individual* from the universe, meaning that ϕ is satisfied in a model by the valuation which assigns the chosen individual to the only free variable of ϕ.

Example 32. Let (M, τ) be the first model from Example 30. Then $(M, \tau) \models \exists x(R(x, x) \wedge P_0(x))$. Indeed, let α be an arbitrary valuation and α' be such that

$$\begin{aligned} \alpha'(x) &= w_1 \\ \alpha'(y) &= \alpha(y) \text{ for } y \neq x \end{aligned}$$

Then $(M, \tau), \alpha' \models R(x, x) \wedge P_0(x)$. Indeed, the latter holds if and only if $(\alpha'(x), \alpha'(x)) \in \tau(R)$ and $\alpha'(x) \in \tau(P_0)$ which is true since $\alpha'(x) = w_1$.

Let now (M, τ) be the second model from the same example. We shall show that $(M, \tau) \models \forall x \exists y \big(R(x, y) \wedge P_3(y)\big)$. Let α be an arbitrary valuation.

Let α' be an valuation which differs from α at most on the value assigned to x. We shall show

$$(M, \tau), \alpha' \models \exists y\big(R(x, y) \wedge P_3(y)\big)$$

So we have to find a way of assigning the value to variable y such that for the resulting valuation α'' we have

$$(M, \tau), \alpha'' \models \big(R(x, y) \wedge P_3(y)\big)$$

The above is true if and only (by the definition of our model!) $\alpha''(x) < \alpha''(y)$ and $3|\alpha''(y)$. So for example we can put $\alpha''(y) = 3(\alpha(x) + 1)$.

It is easy to see that the sentence $\exists x \quad R(x, x)$ is not true in the above model. However this sentence is true in the first model from Example 30, since for example $(w_1, w_1) \in \tau(R)$.

Exercise 10. Let $\sigma = (\mathcal{R}, \mathcal{C}, \rho)$, where $\mathcal{R} = \{R\}$ and $\rho(R) = 2$. Check whether given sentences are true in given models

1. Let $M = \mathbb{N}$ and $(m, n) \in \tau(R)$ if and only if $m < n$.
   (a) $\exists x \forall y R(x, y)$
   (b) $\forall y \exists x R(y, x)$
   (c) $\forall y \exists x R(x, y)$
2. $M = \mathbb{N}$ and $(m, n) \in \tau(R)$ if and only if $m > n$.
   (a) $\exists x \forall y R(x, y)$
   (b) $\forall y \exists x R(y, x)$
   (c) $\forall y \exists x R(x, y)$
3. $M = \mathbb{N}$ and $(m, n) \in \tau(R)$ if and only if m divides n.
   (a) $\exists x \forall y R(x, y)$
   (b) $\exists x \exists y\big(\neg(x = y) \wedge \forall z(\neg R(z, x) \wedge \neg R(z, y)\big)$
4. Let (M, τ) be the first model from Example 30.
   (a) $\forall x \exists y(R(x, y) \wedge P_0(y))$
   (b) $\exists x \exists y(\neg(x = y) \wedge R(x, x) \wedge R(y, y))$
   (c) $\exists x \forall y R(x, y)$.

2.1.1 The standard translation

We will show that the propositional modal logic (pml) is a part of the classical first order logic (cfol). Obviously this is not literally true, since both logic use different symbols. However we can define a translation * of formulae of pml to formulae of cfol. To do this it will be convenient to assume that the only propositional variables we use in formula of modal logic are of the form p_i, where i is a natural number. This translation will be truth preserving in the following sense: if ϕ is a formula of pml, then its translation, denoted ϕ^* will be a formula of cfol over a signature

$$(\{R'\}, \{P_i \mid i \text{ is a natural number }\}), \rho(R) = 2, \rho(P_i) = 1$$

(R will represent the accessibility relation, while $P_i' s$- propositional variables) with precisely one free variable x such that for arbitrary Kripke model $\mathcal{K} = (K, R, f)$ and a world $w \in K$ we will have

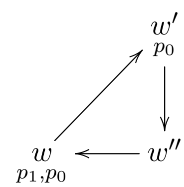
$$\mathcal{K}, w \models \phi \text{ if and only if } \mathcal{K}^*, w \models \phi^*.$$

Where $\mathcal{K}^*$ denotes the natural counterpart of $\mathcal{K}$ in cfol, i.e. the model (K, τ) such that

$$\begin{aligned} \tau(R') &= R \\ \tau(P_i) &= \{w \in K \mid \mathcal{K}, w \models p_i\} \end{aligned}$$

Let us unravel this definition: $\mathcal{K}^*$ results from $\mathcal{K}$ by taking possible worlds (i.e. elements of K) to be individuals (i.e. the universe of Kripke model form now the universe of first order model) and interpreting R' as the accessibility relation. Moreover each P_i is interpreted as the set of those worlds w which are labelled with p_i in $\mathcal{K}$.

Example 33. Let $\mathcal{K} = (K, R, f)$ be the following model



then $\mathcal{K}^* = (K, \tau)$ and

$$\begin{aligned} \tau(R') &= \{(w, w'), (w', w''), (w'', w)\}, \\ \tau(P_0) &= \{w, w'\}, \\ \tau(P_1) &= \{w\}, \\ \tau(P_i) &= \emptyset, \text{ for } i > 1. \end{aligned}$$

To define the translation it will be convenient to assume that the individual variables we use are numbered with natural numbers: we will denote them by $x_0, x_1, x_2 \ldots$. Let us finally define *. The translation will be *compositional*, i.e. the translation of a formula will be fully determined by its main connective (or modal operator) and the translations of its immediate subformulae. We will define it by induction on the structure of a formula: we start with the simplest formulae possible, i.e. propositional variables, and then show how to compute the translations of more complex formulae once the translations of their immediate subformulae have been fixed.

Definition 34 (The standard translation)**.** For every p_i,

$$(p_i)^* = P_i(x_0).$$

If the translation of ψ has already been defined, then

$$(\neg\psi)^* = \neg(\psi)^*.$$

If the translations of ψ, θ have already been defined, then

$$\begin{array}{rcl}(\theta \wedge \psi)^* & = & (\theta)^* \wedge (\psi)^* \\ (\theta \vee \psi)^* & = & (\theta)^* \vee (\psi)^* \\ (\theta \to \psi)^* & = & (\theta)^* \to (\psi)^*\end{array}$$

Suppose now the translation of ψ has been defined and let n be the number of modal operators used in ψ. Then

$$(\Diamond\psi)^* = \exists x_{n+1}\big(R(x_0, x_{n+1}) \wedge (\psi)^*[x_{n+1}/x_0]\big)$$
$$(\Box\psi)^* = \forall x_{n+1}\big(R(x_0, x_{n+1}) \to (\psi)^*[x_{n+1}/x_0]\big)$$

where $\phi[x_{n+1}/x_0]$ denotes the result of replacing in ϕ every occurrence of variable x_0 with variable x_{n+1}.

It would be best to see how this definition works on a concrete example:

Example 35. We will compute the translation of $\Box\Diamond p_0 \to \Diamond\Box p_0$. The main connective in this formula is the implication, so

$$(\Box\Diamond p_0 \to \Diamond\Box p_0)^* = (\Box\Diamond p_0)^* \to (\Diamond\Box p_0)^*$$

Let us compute $(\Box\Diamond p_0)^*$ first. The main connective is $\Box$ and $\Diamond p_0$ contains exactly one modal operator. So

$$(\Box\Diamond p_0)^* = \forall x_2\big(R(x_0, x_2) \to (\Diamond p_0)^*[x_2/x_0]\big).$$

So let us compute $(\Diamond p)^*$: p contains no modal operators, so

$$\begin{aligned}(\Diamond p_0)^* &= \exists x_1\big(R(x_0,x_1) \wedge (p_0)^*[x_1/x_0]\big)\\ &= \exists x_1\big(R(x_0,x_1) \wedge (P_0(x_0))[x_1/x_0]\big)\\ &= \exists x_1\big(R(x_0,x_1) \wedge P_0(x_1)\big)\end{aligned}$$

Now we have to plug in the last formula changing x_0 to x_2. We get:

$$(\Box\Diamond p_0)^* = \forall x_2\big(R(x_0,x_2) \to \exists x_1\big(R(x_2,x_1) \wedge P_0(x_1)\big)\big).$$

Similarly

$$(\Diamond\Box p_0)^* = \exists x_2\big(R(x_0,x_2) \wedge \forall x_1\big(R(x_2,x_1) \to P_0(x_1)\big)\big)$$

So the translation of $\Box\Diamond p_0 \to \Diamond\Box p_0$ is the following formula:

$$\Big(\forall x_2\big(R(x_0,x_2) \to \exists x_1\big(R(x_2,x_1) \wedge P_0(x_1)\big)\big)\Big) \longrightarrow$$
$$\longrightarrow \Big(\exists x_2\big(R(x_0,x_2) \wedge \forall x_1\big(R(x_2,x_1) \to P_0(x_1)\big)\big)\Big)$$

Exercise 11. Compute translations of the following formulae:

1. $\Box p_3$
2. $\Diamond(p_1 \wedge p_{17})$
3. $\Box(p_0 \vee \neg\Diamond p_{13})$
4. $\Box\neg p_2 \vee \Box p_2$
5. $\Box\Diamond p_0 \to \Diamond\Box p_0$
6. $\Box(\Box p_0 \to p_0) \to \Box p_0$
7. $\Box(p_3 \wedge p_2) \to \big(\neg\Diamond\Box P_5\big)$.
8. $\Box(p_4 \vee \neg\neg\Diamond p_3)$.

3 Quantified modal logic

3.1 Models for the quantified modal logic

We extend the definition of first-order formulae in a natural way, so that our formulae may contain some modal operators. Let σ be a relational signature and V an infinite set of *variables*.The set of relational *first-order modal formulae* is the smallest set F satisfying the following conditions:

1. $R_i(v_1, \ldots, v_n) \in F$, where v_i are variables and R_i is some relation symbol from σ of arity n.

2. $\phi \wedge \psi \in F$, when $\phi \in F$ and $\psi \in F$.

3. $\phi \vee \psi \in F$, when $\phi \in F$ and $\psi \in F$.

4. $\neg\phi \in F$, when $\phi \in F$.

5. $\forall v \;\; \phi \in F$, when $\phi \in F, v \in V$.

6. $\exists v \;\; \phi \in F$, when $\phi \in F, v \in V$.

7. $\Box\phi \in F$, when $\phi \in F$.

8. $\Diamond\phi \in F$, when $\phi \in F$.

As before, unwinding this definition, we see that quantified modal formulae look like the regular first-order formulae, but we allow some extra operators $\Box$ and $\Diamond$.

Now we are ready to introduce the models for the first order modal logic.

Definition 36. By a Kripke model $\mathcal{W}$ for first-order modal logic over the signature σ we mean the structure (W, R, D, V) such that:

1. W is a nonempty set whose elements we call **possible worlds** like in the propositional modal case.

2. R is a binary relation on W, which we call the **accessibility relation** as in the propositional case.

3. D is a function with domain W such that for every $w \in W$, $D(w)$ is a nonempty set. We think of $D(w)$ as of the domain of the possible world associated with w. The set theoretical sum of the set of values of D will be called **the set of individuals** of $\mathcal{W}$. I.e. the set of individuals of $\mathcal{W}$ is

$$\bigcup_{w \in W} D(w)$$

4. V is a function which takes a world w and a symbol h from the signature and returns the interpretation of h in $D(w)$.

Remark 37. If (W, R, D, V) is a Kripke model for FOML, then for every $w \in W$ $(D(w), V(w, \cdot))$ is a model for first order logic, as in Definition 29. ($V(w, \cdot)$ denotes the function resulting from V by fixing one of its arguments)

Again, we intuitively think of the elements $w \in W$ as of the possible worlds. As before, the relation R holds between w and v if, intuitively, world v is possible *from the point of view* of world w.

Definition 38. Let $\mathcal{W} = (W, R, D, V)$ be a FOML model. A **valuation** α is any function from the chosen set of first order variables to the set of individuals of $\mathcal{W}$. For every $w \in W$ and every FOML formula ϕ we call α a w-**valuation with respect to** ϕ if for every free variable v in ϕ, $\alpha(v)$ belongs to $D(w)$.

Now, we define what is a satisfaction of a formula in a given model for FOML.

Definition 39. Let $\mathcal{W} = (W, R, D, V)$ be any Kripke model over a signature σ. We define what does it mean for a formula ϕ to be satisfied in a world $w \in W$ in the model $\mathcal{W}$ under a valuation α by induction on complexity of formulae.

1. If $\phi = R(x_1, \ldots, x_n)$, then $\mathcal{W}, w, \alpha \models \phi$ iff $(\alpha(x_1), \ldots, \alpha(x_n)) \in V(w, R)$
2. $\mathcal{W}, w, \alpha \models (\phi \wedge \psi)$ iff $\mathcal{W}, w, \alpha \models \phi$ and $\mathcal{W}, w, \alpha \models \psi$.
3. $\mathcal{W}, w, \alpha \models (\phi \vee \psi)$ iff $\mathcal{W}, w, \alpha \models \phi$ or $\mathcal{W}, w, \alpha \models \psi$.
4. $\mathcal{W}, w, \alpha \models \neg\phi$ iff it is not the case that $\mathcal{W}, w, \alpha \models \phi$.
5. $\mathcal{W}, w, \alpha \models \exists v\, \phi$ iff there exists a w-valuation with respect to ϕ α' which may differ from α at most in what it ascribes to variable v such that $\mathcal{W}, w, \alpha' \models \phi$.
6. $\mathcal{W}, w, \alpha \models \forall v\ \phi$ iff $\mathcal{W}, w, \alpha' \models \phi$ for all w-valuations with respect to ϕ α' which differ from α at most in what they ascribe to variable v.
7. $\mathcal{W}, w, \alpha \models \Box\phi$ iff $\mathcal{W}, v, \alpha \models \phi$ for all worlds v such that $R(w, w')$ and α is a w' valuation with respect to ϕ.
8. $\mathcal{W}, w, \alpha \models \Diamond\phi$ iff $\mathcal{W}, v, \alpha \models \phi$ for some world v such that $R(w, w')$ and α is a v valuation with respect to ϕ.

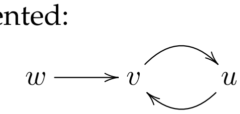
Similarly to the first-order case, we say that a formula over a signature σ is true in a Kripke model, at a given world w iff it is satisfied there under all valuations. We say that it is true in a model $\mathcal{W}$, if it is true at all worlds in $\mathcal{W}$, we say that it is valid, if it is true in all models over the signature σ at all worlds.

Exercise 12. Check whether the following formulae hold in every Kripke model.

1. $\big(\exists x \Diamond P(x)\big) \to \exists x P(x)$.
2. $\big(\Diamond \exists x P(x)\big) \to \big(\exists x \Diamond P(x)\big)$.
3. $\big(\forall x \Box P(x)\big) \to \big(\Box \forall x P(x)\big)$.
4. $\big(\Diamond \forall x P(x)\big) \to \big(\forall x \Diamond P(x)\big)$.
5. $\big(\Diamond \forall x \Box P(x)\big) \to \big(\Diamond \Box \forall x P(x)\big)$.
6. $\big(\exists x \Box P(x)\big) \to \big(\Box \exists x P(x)\big)$.
7. $\big(\Diamond \forall x \exists y (Q(x,y))\big) \to \big(\forall x \Diamond \exists y Q(x,y)\big)$
8. $\big(\Box \exists x \forall y Q(x,y)\big) \to \big(\exists x \Box \forall y Q(x,y)\big)$
9. $\Big(\big(\exists x \Diamond P(x)\big) \wedge \big(\Box \forall x (P(x) \to F(x))\big)\Big) \to \big(\exists x \Diamond F(x)\big)$
10. $\big(\Box \forall x (P(x) \to F(x)) \wedge \Box \forall x (F(x) \to G(x))\big) \to \forall x (\Box P(x) \to \Box G(x))$
11. $\exists x \exists y (Q(x,y) \wedge \Box Q(y,x))$

Exercise 13. Check whether the following formulae hold in the given models:

1. Let W and R be as presented:



   (a) Define the Domain and the interpretation functions as

$$D(w) = \{a,b,c\}, D(v) = \{a,b\}, D(u) = \{b,c\}$$

$$V(P,w) = \{a,b,c\}, V(P,v) = \{a,b\}, V(P,u) = \{b\}.$$

   Verify whether the following formulae are satisfied in $(W,R,D,V), w$

i. $\Box\forall x P(x)$
ii. $\forall x \Box P(x)$
iii. $\forall x \Box\Box P(x)$
iv. $\exists x(P(x) \wedge \Diamond\neg P(x))$
v. $\exists x(P(x) \wedge \Diamond\Diamond\neg P(x))$

(b) Define the domain and the interpretation functions as

$$D(w) = \{a, b\}, D(v) = \{a, b, c\}, D(u) = \{b, c\}$$

$$V(P, w) = \{a, b\}, V(P, v) = \{a, b\}, V(P, u) = \{b\}.$$

Verify whether the following formulae are satisfied in $(W, R, D, V), w$

i. $\Diamond\exists x \Diamond\neg P(x)$
ii. $\forall x \Box P(x) \to \Box\forall x \Box P(x)$

(c) Define the Domain and the interpretation functions as

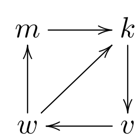
$$D(w) = \{a\}, D(v) = \{a, b\}, D(u) = \{a, b, c\}$$

$$V(P, w) = \{a, b, c\}, V(P, v) = \emptyset, V(P, u) = \{b\}.$$

Verify whether the following formulae are satisfied in $(W, R, D, V), u$ (mind that we switched the world from which we start)

i. $\exists x \Box P(x) \to \Box\exists x P(x)$.

2. Let the domain D and the relation R be now as follows:



Define the interpretation function D and the interpretation function V as follows:

(a)

$$D(w) = \{a, b\}, D(m) = \{a\}, D(k) = \{b\}, D(v) = \{a, b, c\}$$

$$V(Q, w) = \{(a, b), (b, a)\}, V(Q, m) = \{(a, a)\},$$
$$V(Q, k) = \{(b, b)\}, V(Q, v) = \{(a, b), (a, c), (b, c)\}$$

Verify whether the following formulae holds in $(W, R, D, V), w$

i. $\forall x \Box Q(x,x) \to \forall x Q(x,x)$
ii. $\forall x \Box Q(x,x) \to \Box \Diamond \forall x \exists y Q(x,y)$

(b)

$$D(w) = \{a,b\}, D(m) = \{a\}, D(k) = \{b,c\}, D(v) = \{a,b,c\}$$

$$V(Q,w) = \{(a,b),(b,a)\}, V(Q,m) = \{(a,a)\},$$
$$V(Q,k) = \{(b,c)\}, V(Q,v) = \{(a,b),(a,c),(b,c)\}$$

Verify whether the following formulae holds in $(W,R,D,V),w$

i. $\exists x \Box Q(x,x)$
ii. $\forall x \Box Q(x,x)$
iii. $\big(\forall x \forall y (Q(x,y) \equiv Q(y,x))\big) \to \Diamond\big(\forall x \forall y (Q(x,y) \equiv Q(y,x))\big)$
iv. $\big(\forall x \forall y (Q(x,y) \equiv Q(y,x))\big) \to \Box\big(\forall x \forall y (Q(x,y) \equiv Q(y,x))\big)$

Exercise 14. Solve the following puzzle of Quine:

> Cyclists are necessarily two-legged, but not necessarily rational. Mathematicians are necessarily rational, but not necessarily two-legged. Consider a cycling mathematician. Is he both necessarily rational and not necessarily rational, with the same contradiction in his legs?

Quotation after "Modal Logic for Open Minds".

4 Intuitionistic logic

Intuitionistic logic aims to capture patterns of constructive reasoning. In a way, we assume that nothing is either true or false, until proven to be so. In particular we do not assume that the law of excluded middle holds.

4.1 Propositional intuitionistic logic

The syntax of propositional intuitionistic logic is almost the same as the syntax of the classical propositional logic. We take $\vee$, $\wedge$, $\to$ and $\bot$ as primitive symbols, where this last one is a propositional constant. For example

$$(p \to \bot) \wedge (q \wedge (r \vee \bot))$$

is a well-built formula. The semantics for Intuitionistic Logic is completely different. We will translate intuitionistic formulae into some specific class of modal formulae and then define satisfaction of the intuitionistic formulae in Kripke models via this translation.

Definition 40. We define a translation of a propositional intuitionistic formula ϕ to a modal formula ϕ^T by induction on complexity in the following way:

1. $\phi^T = \phi$ for ϕ propositional variables.
2. $(\phi \wedge \psi)^T = \phi^T \wedge \psi^T$.
3. $(\phi \vee \psi)^T = \phi^T \vee \psi^T$.
4. $(\phi \to \psi)^T = \Box(\phi^T \to \psi^T)$.
5. $(\bot)^T = (p \wedge \neg p)$

Definition 41 (Negation)**.** We treat $\neg$ as defined symbol. We put

$$\neg\phi := \phi \to \bot$$

Now we define the class of models for the intuitionistic propositional logic. It will consist of Kripke models of particular kind.

Definition 42. Let $\mathcal{K} = (K, R, V)$ be a Kripke model for the propositional modal logic. We call M a model for the propositional intuitionistic logic iff the following additional conditions are satisfied:

1. R is a partial order, i.e. it is antisymmetric, reflexive and transitive.
2. V is monotonous, i.e. for every w, v such that wRv, if $\mathcal{K}, w \models p$, then $\mathcal{K}, v \models p$ for every propositional variable p.

Now we define, what does it mean for an intuitionistic propositional formula to be defined in a Kripke model.

Definition 43. Let ϕ be an intuitionistic propositional formula, let $\mathcal{K}$ be a Kripke model for the propositional intuitionistic logic and let w be a world in K. We say that ϕ is *satisfied* in the model $\mathcal{K}$ at the world w, $\mathcal{K}, w \models_i \phi$ iff

$$\mathcal{K}, w \models \phi^T$$

where $\models$ is a satisfaction relation for Propositional Modal Logic.

Remark 44 (Satisfaction conditions for Propositional Intuitionistic Logic)**.** Let us note that $\models_i$ satisfies:

1. $\mathcal{K}, w \models_i p$ iff $\mathcal{K}, w \models p$
2. $\mathcal{K}, w \models_i \phi \vee \psi$ iff $\mathcal{K}, w \models_i \phi$ or $\mathcal{K}, w \models_i \psi$
3. $\mathcal{K}, w \models_i \phi \wedge \psi$ iff $\mathcal{K}, w \models_i \phi$ and $\mathcal{K}, w \models_i \psi$
4. $\mathcal{K}, w \models_i \phi \rightarrow \psi$ iff for every v such that wRv, if $\mathcal{K}, w \models_i \phi$, then $\mathcal{K}, w \models_i \psi$
5. it is never true that $\mathcal{K}, w \models_i \bot$

From the last two it holds that

$$\mathcal{K}, w \models_i \neg\phi \text{ iff for every } v \text{ such that } wRv, \mathcal{K}, v \not\models_i \phi$$

Definition 45. We say that a formula ϕ *holds* in a model $\mathcal{K} = \langle K, R, V \rangle$, $\mathcal{K} \models_i \phi$, if for every $w \in K$ it holds that $\mathcal{K}, w \models_i \phi$. We say that a formula ϕ is *intuitionistically valid* if for every $\mathcal{K}$ ϕ holds in $\mathcal{K}$. We use $\models_i \phi$ to denote that ϕ is valid.

Convention 46. When drawing models, to avoid drawing many arrows, we implicitly assume that the relation is reflexive and transitive. Hence every time we have

$$w \longrightarrow v \longrightarrow x$$

we mean that also the following pairs are in the accessibility relation:

1. $\langle w, w \rangle$
2. $\langle v, v \rangle$
3. $\langle x, x \rangle$
4. $\langle w, x \rangle$

Exercise 15. Verify whether given models satisfy given formulae of intuitionistic propositional logic:

1. Model (p, q denote propositional variables and w, v, x denote worlds)

$$\underset{p}{w} \longrightarrow \underset{p,q}{v} \longrightarrow \underset{p,q}{x}$$

Check whether

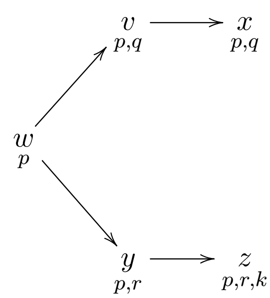
(a) $\mathcal{K}, w \models_i p \to q$

(b) $\mathcal{K}, w \models_i \neg p$

(c) $\mathcal{K}, w \models_i \neg q$

(d) $\mathcal{K}, w \models_i \neg\neg q$

2. Model: (p, q, r, k denote propositional variables and w, v, x, y, z denote worlds)



Formulae:

(a) $\mathcal{K}, w \models_i r \vee \neg r$

(b) $\mathcal{K}, w \models_i q \to p$

(c) $\mathcal{K}, w \models_i \neg q$

(d) $\mathcal{K}, y \models_i \neg q$

(e) $\mathcal{K}, v \models_i p \to q$

(f) $\mathcal{K}, w \models_i (p \to q) \to q$

Exercise 16. Check whether the following formulae of propositional intuitionistic logic are intuitionistically valid:

1. $p \to p$
2. $(p \to q) \to p$
3. $p \to (\neg\neg p)$
4. $(\neg\neg p) \to p$
5. $(p \wedge q) \to p$
6. $(p \wedge \neg p) \to q$

4.1.1 Hilbert-style proof system for IPL and the Disjunction Property

Let us define the Hilbert-style proof system for Intuitionistic Propositional Logic. The notion of *proof* and *provability* is the same as in the classical case, see Definition 5 and we take Modus Ponens as our unique rule of reasoning. In contrast to classical case we take the following axiom schemes (i.e. every formula of one of the following shapes is our axiom):

1. $\phi \to (\psi \to \phi)$
2. $(\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\phi \to \chi))$
3. $\phi \to \phi \vee \psi$
4. $\psi \to \phi \vee \psi$
5. $(\phi \to \chi) \to ((\psi \to \chi) \to (\phi \vee \psi \to \chi))$
6. $\phi \wedge \psi \to \psi$
7. $\phi \wedge \psi \to \phi$
8. $\phi \to (\psi \to \phi \wedge \psi)$
9. $\bot \to \phi$

The above system is taken from Proof Theory lecture notes by Benno van den Berg (available here). Let us use $\vdash_i \phi$ to denote the fact that ϕ is provable in the above defined system. Then we have

Theorem 47 (Completeness for IPL)**.** *For every formula ϕ, $\vdash_i \phi$ if and only if $\models_i \phi$.*

On of the distinctive virtues of intuitionistic logic, which witness its constructive character, is the following Disjunction Property:

Theorem 48 (Disjunction Property)**.** *For every formulae ϕ, ψ we have*

$$\vdash_i \phi \vee \psi \text{ if and only if } \vdash_i \phi \text{ or } \vdash_i \psi$$

We shall prove the above using the following monotonicity condition for intuitionistic Kripke models: it generalizes condition 2 in Definition 42 to arbitrary formulae:

Lemma 49. *Let ϕ be an intuitionistic formula, $\mathcal{K} = \langle K, R, V \rangle$ an intuitionistic Kripke model. Then we have: for all $w, v \in K$ such that wRv*

$$\mathcal{K}, w \models \phi \text{ implies } \mathcal{K}, v \models \phi \qquad \text{(MON)}$$

Proof. We use induction on the complexity of formulae.

Base step If p is a propositional variable, then for every $w, v \in K$ such that wRv, MON holds by the definition of Intuitionistic Kripke Model.

Induction step Suppose ϕ is a compound formula and let us distinguish cases.

Case 1 Suppose $\phi = \psi_0 \vee \psi_1$ and MON holds for ψ_0 and ψ_1 and arbitrary $w, v \in K$ such that wRv. Let us fix $w, v \in K$ such that wRv and suppose

$$\mathcal{K}, w \models \psi_0 \vee \psi_1$$

then by definition either $\mathcal{K}, w \models \psi_0$ or $\mathcal{K}, w \models \psi_1$. Without loss of generality suppose the former holds (if the latter holds, then the proof is the same). By induction assumption for ψ_0 we get

$$\mathcal{K}, v \models \psi_0$$

Hence $\mathcal{K}, v \models \psi_0 \vee \psi_1$ and this step is finished.

Case 2 Suppose $\phi = \psi_0 \wedge \psi_1$. The proof is as above and we leave it as an exercise.

Case 3 Suppose $\phi = \psi_0 \to \psi_1$ and MON holds for ψ_0, ψ_1 and arbitrary w, v such that wRv. Let us fix $w, v \in K$ such that wRv. Assume that

$$\mathcal{K}, w \models \psi_0 \to \psi_1.$$

We have to check whether $\mathcal{K}, v \models \psi_0 \to \psi_1$. This amounts to checking whether for arbitrary u such that vRu

$$\text{if } \mathcal{K}, u \models \psi_0 \text{ then } \mathcal{K}, u \models \psi_1. \qquad (*)$$

So let us fix arbitrary u such that vRu. We have wRv and vRu, hence, by transitivity wRu. Hence $*$ follows by our assumption that $\mathcal{K}, w \models \psi_0 \to \psi_1$. □

Now we may proceed to the proof of Theorem 48:

Proof of Theorem 48. Let us assume that $\nvdash_i \phi$ and $\nvdash_i \psi$. By Theorem 47 there exists $\mathcal{K} = \langle K, R, V \rangle$, $w \in K$ such that

$$\mathcal{K}, w \not\models \phi$$

and $\mathcal{K}' = \langle K', R', V \rangle$ and $w' \in K'$ such that

$$\mathcal{K}', w' \not\models \psi.$$

Without loss of generality assume that $K \cap K' = \emptyset$ and that $w'' \notin K \cup K'$. Let us define $\mathcal{K}'' = \langle K'', R'', V'' \rangle$ by

$$\begin{array}{rcl} K'' & = & \{w''\} \cup K \cup K' \\ R'' & = & \{\langle w'', w \rangle, \langle w'', w' \rangle\} \cup R \cup R' \\ V'' & = & V \cup V' \end{array}$$

($\mathcal{K}''$ results from "glueing" together models $\mathcal{K}$ and $\mathcal{K}'$ using the world w''). Let us observe that for every formula θ we have

$$\mathcal{K}'', w \models \theta \iff \mathcal{K}, w \models \theta \qquad (*)$$

and

$$\mathcal{K}'', w' \models \theta \iff \mathcal{K}', w' \models \theta. \qquad (**)$$

We claim that $\mathcal{K}'', w'' \not\models \phi \vee \psi$, which would end our proof. Indeed, for suppose the contrary. Without loss of generality assume that $\mathcal{K}'', w'' \models \phi$. Then by Lemma 49 we have that $\mathcal{K}'', w \models \phi$, and by $*$

$$\mathcal{K}, w \models \phi$$

which contradicts our assumption. $\square$

4.2 Intuitionistic First-Order Logic

The syntax of Intuitionistic First-Order Logic is the same as the syntax of Classical First Order Logic, except for we take all of $\wedge$, $\vee$, $\rightarrow$, $\bot$, $\forall$ and $\exists$ as primitive symbols. Let us define the semantics for this logic: we start with the definition of a submodel.

Definition 50 (Submodel)**.** Let $\mathcal{M} = (M, \tau_M)$ and $\mathcal{N} = (N, \tau_N)$ be two first order models over constant-relational signature σ in the sense of Definition 29. We say that $\mathcal{M}$ is a *submodel* of $\mathcal{N}$ if and only if

1. for every relation $R \in \sigma$, $\tau_M(R) \subseteq \tau_N(R)$,

2. for every constant $c \in \sigma$, $\tau_M(c) = \tau_N(c)$.

If $\mathcal{M}$ is a submodel of $\mathcal{N}$ then we shall denote it by $\mathcal{M} \subseteq \mathcal{N}$.

Example 51. Let σ consists of a unary predicate P and a constant c. Let

$$\begin{aligned} M &= \{x, y\}, & N &= \{x, y, z\} \\ \tau_M(P) &= \{x\}, & \tau_N(P) &= \{x, y, z\} \\ \tau_M(c) &= x, & \tau_N(c) &= x \end{aligned}$$

Then $\mathcal{M} = (M, \tau_{\mathcal{M}})$ is a submodel of $\mathcal{N} = (N, \tau_{\mathcal{N}})$.

If we altered this definition putting $\tau'_M(P) = \{x, y\}$ and $\tau'_N(P) = \{x, z\}$, then (M, τ'_M) would not be a submodel of (N, τ'_N).

As in the case of propositional logic we shall define models for Intuitionistic First-Order Logic as particular Kripke models for First-Order Modal Logic. It will be convenient to use one convention

Convention 52. In Definition **??** we defined models as for First-Order Modal Logic as quadruples $\langle W, R, D, V\rangle$ such that D is the domain function and V is the interpretation function, such that for each $w \in W$,

$$\langle D(w), V(w, \cdot)\rangle$$

is a model for First-Order Logic ($V(w, \cdot)$ denotes the function of one argument resulting from V by fixing one particular world w). Equivalently we can say that we have a family $\{\mathcal{A}_w\}_{w\in W}$ of models for First-Order Logic parametrized by elements of W and define First Order Kripke models as

$$\langle W, R, \{\mathcal{A}_w\}_{w\in W}\rangle$$

where,as previously, W is a non-empty set, R is a binary relation ("accessibility" relation) and $\{\mathcal{A}_w\}_{w\in W}$ is a family of models for First-Order Logic parametrized by elements of W. We will use this definition.

Definition 53. An Kripke models for Intuitionistic First Order Logic is a triple $\langle W, R, \{A\}_{w\in W}\rangle$ where

1. W is a non-empty set.
2. $R \subseteq W^2$ is a partial order.
3. $\langle W, R, \{\mathcal{A}\}_{w\in W}\rangle$ is a family of models for First Order Logic over the same constant-relational signature σ such that for every $w, v \in W$ such that wRv we have
$$\mathcal{A}_w \subseteq \mathcal{A}_v$$

Remark 54. Let $\mathcal{W} = \langle W, R, \{\mathcal{A}_w\}_{w\in W}\rangle$ be a Kripke model for First-Order Intuitionistic Logic, $w \in W$ and let ϕ be a formula (of First-Order Logic). Recall the notion of w-valuation with respect to ϕ introduced in Definition 38. Observe that for every v such that wRv, if α is a w-valuation with respect to ϕ, then it is also v-valuation with respect to ϕ.

Convention 55. Let α be any valuation and x - a variable. For every b, by $\alpha[x \mapsto b]$ we denote the unique valuation β defined

$$\begin{aligned}\beta(y) &= \alpha(y) \text{ for } y \neq x\\ \beta(x) &= b\end{aligned}$$

I.e. $\alpha[x \mapsto b]$ differs from α at most on the value assigned to x, and $\alpha[x \to b]$ assigns b to x.

Definition 56 (Satisfaction Relation)**.** Let $\mathcal{W} = \langle W, R, \{\mathcal{A}_w\}_{w\in W}\rangle$ be a Kripke model for First-Order Intuitionistic Logic. For every $v \in W$ let $\mathcal{A}_v = \langle A_v, \tau_v\rangle$. By induction on the complexity of first order formula ϕ we define the relation

$$\mathcal{W}, w, \alpha \models_i \phi$$

where $w \in W$ and α is a w valuation with respect to ϕ.

1. if $\phi = P(x_{i_0}, \ldots, x_{i_n})$, where x_{i_k} are either free variables or constants and P is an n-ary relational symbol from the signature, then

   $$\mathcal{W}, w, \alpha \models_i P(x_{i_0}, \ldots, x_{i_n})$$

   iff $\langle a_{i_0}, \ldots, a_{i_n}\rangle \in \tau_w(P)$ where for each $k \leq n$,

   $$a_{i_k} = \begin{cases} \tau_w(x_{i_k}), & \text{if } x_{i_k} \text{ is a constant,}\\ \alpha(x_{i_k}), & \text{if } x_{i_k} \text{ is a variable}\end{cases}$$

2. similarly if ϕ is of the form $x_{i_0} = x_{i_1}$ then

   $$\mathcal{W}, w, \alpha \models_i x_{i_0} = x_n$$

   iff $a_0 = a_1$ where a_l (for $l \leq 1$) equals $\tau_w(x_{i_l})$ iff x_{i_l} is a constant and $\alpha(x_{i_l})$ iff x_{i_l} is a variable.

3. if $\phi = \psi_0 \wedge \psi_1$, then

   $$\mathcal{W}, w, \alpha \models_i \psi_0 \wedge \psi_1$$

   iff $\mathcal{W}, w, \alpha \models_i \psi_0$ and $\mathcal{W}, w, \alpha \models_i \psi_1$.

4. if $\phi = \psi_0 \vee \psi_1$, then
$$\mathcal{W}, w, \alpha \models_i \psi_0 \vee \psi_1$$
iff $\mathcal{W}, w, \alpha \models_i \psi_0$ or $\mathcal{W}, w, \alpha \models_i \psi_1$.

5. if $\phi = \psi_0 \to \psi_1$, then
$$\mathcal{W}, w, \alpha \models_i \psi_0 \to \psi_1$$
iff for every v such that wRv, if $\mathcal{W}, w, \alpha \models_i \psi_0$, then $\mathcal{W}, w, \alpha \models_i \psi_1$.

6. if $\phi = \exists x \psi$, for some variable x, then
$$\mathcal{W}, w, \alpha \models_i \exists x \psi$$
iff there exists $a \in A_w$ such that $\mathcal{W}, w, \alpha[x \mapsto a] \models_i \psi$.

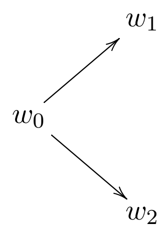
7. if $\phi = \forall x \psi$, for some variable x, then
$$\mathcal{W}, w, \alpha \models_i \forall x \psi$$
iff for all v such that wRv and all $a \in A_v$, $\mathcal{W}, w, \alpha[x \mapsto a] \models_i \psi$

As usual, if ϕ is a sentence, then we define
$$\mathcal{W}, w \models_i \phi$$
iff for every valuation α
$$\mathcal{W}, w, \alpha \models_i \phi$$

Convention 57. If $\mathcal{W} = \langle W, R, \{\mathcal{A}_w\}_{w \in W} \rangle$ is a Kripke model for First-Order Intuitionistic Logic then for every $w \in W$ the universe of $\mathcal{A}_w$ will be denoted with A_w and the interpretation function with τ_{A_w}.

Remark 58. Let us observe that, for every Kripke model $\mathcal{W} = \langle W, R, \{\mathcal{A}_w\}_{w \in W} \rangle$ and every valuation α
$$\mathcal{W}, w\alpha[x \mapsto a] \models_i P(x)$$
iff $a \in \tau_w(P)$, where $\mathcal{A}_w = \langle A_w, \tau_w \rangle$.

Example 59. Let $W = \{w_0, w_1, w_2\}$ and $R = \{\langle w_0, w_1\rangle, \langle w_0, w_2\rangle, \langle w_0, w_0\rangle, \langle w_1, w_1\rangle, \langle w_2, w_2\rangle\}$. Hence W and R might be depicted



Let us work over signature with one unary predicate P. Define $A_{w_0} = \{a, b\}$, $A_{w_1} = \{a, b, c\}$, $A_{w_2} = \{a, b, d\}$ and

$$\begin{aligned} \tau_{w_0}(P) &= \{a, b\} \\ \tau_{w_1}(P) &= \{a, b, c\} \\ \tau_{w_2}(P) &= \{a, b\} \end{aligned} \tag{1}$$

Finally let $\mathcal{A}_{w_0} = \langle A_{w_0}, \tau_{w_0}\rangle$, $\mathcal{A}_{w_1} = \langle A_{w_1}, \tau_{w_1}\rangle$, $\mathcal{A}_{w_2} = \langle A_{w_2}, \tau_{w_2}\rangle$, and

$$\mathcal{W} = \langle W, R, \{\mathcal{A}_w\}_{w \in W}\rangle.$$

Then $\mathcal{W}, w_0 \not\models_i \forall x P(x)$. Indeed let α be any valuation. Then, unravelling the definition we have:

$$\mathcal{W}, w_0, \alpha \models_i \forall x P(x)$$

iff

$$\text{for all } v \text{ such that } w_0 R v \text{ and all } a \in A_v\ \mathcal{W}, v, \alpha[x \mapsto a] \models_i P(x)$$

The above condition holds if and only if each of the three conditions below is fulfilled

1. $A_{w_0} = \tau_{w_0}(P)$
2. $A_{w_1} = \tau_{w_1}(P)$
3. $A_{w_2} = \tau_{w_2}(P)$

The last condition however does not hold, since $d \in A_{w_2} \setminus \tau_{w_2}(P)$.