TOPOLOGICAL MODELS OF ARITHMETIC

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ABSTRACT. Ali Enayat had asked whether there is a nonstandard model of Peano arithmetic (PA) that can be represented as $\langle \mathbb{Q}, \oplus, \otimes \rangle$, where $\oplus$ and $\otimes$ are continuous functions on the rationals $\mathbb{Q}$. We prove, affirmatively, that indeed every countable model of PA has such a continuous presentation on the rationals. More generally, we investigate the topological spaces that arise as such topological models of arithmetic. The reals $\mathbb{R}$, the reals in any finite dimension $\mathbb{R}^n$, the long line and the Cantor space do not, and neither does any Suslin line; many other spaces do; the status of the Baire space is open.

1. INTRODUCTION

The first author had inquired whether a nonstandard model of arithmetic could be continuously presented on the rational numbers.

Main Question 1 (Enayat [Ena09]). *Are there continuous functions $\oplus$ and $\otimes$ on the rational numbers $\mathbb{Q}$, such that $\langle \mathbb{Q}, \oplus, \otimes \rangle$ is a nonstandard model of arithmetic?*

By a model of arithmetic, what we mean here is a model of the first-order Peano axioms PA, although we shall also consider various weakenings of this theory. The theory PA asserts of a structure $\langle M, +, \cdot \rangle$ that it is the non-negative part of a discretely ordered ring, plus the induction principle for assertions in the language of arithmetic. The natural numbers $\langle \mathbb{N}, +, \cdot \rangle$, for example, form what is known as the *standard* model of PA, but there are also many nonstandard models, including continuum many non-isomorphic countable models.

We shall answer the question affirmatively, and indeed, our main theorem (theorem 5) shows that every countable model of PA is continuously presented on $\mathbb{Q}$. We define generally that a *topological* model of arithmetic is a topological space X equipped with continuous functions $\oplus$ and $\otimes$, for which $\langle X, \oplus, \otimes \rangle$ satisfies the desired arithmetic theory. In such a case, we shall say that the underlying space X continuously *supports* a model of arithmetic and that the model is continuously *presented* upon the space X.

Question 2. *Which topological spaces support a topological model of arithmetic?*

In section 3, we shall prove that the reals $\mathbb{R}$, the reals in any finite dimension $\mathbb{R}^n$, the long line and the Cantor space do not support a topological model of arithmetic, and neither does any Suslin line. Meanwhile, there are many other spaces that do support topological models, including many uncountable subspaces of the plane $\mathbb{R}^2$. It remains an open question whether any uncountable Polish space, including the Baire space, can support a topological model of arithmetic.

In light of our answer to the main question in the case of $\mathbb{Q}$, where we showed not only that some models of PA are continuously presented on $\mathbb{Q}$, but that all countable models of PA can

be continuously presented on $\mathbb{Q}$—and the same is true of the indiscrete and discrete spaces—it seems natural to ask whether this none/all phenomenon continues with other spaces.

Question 3. *If a topological space X continuously supports a model of* PA*, then does X continuously support a copy of every model of* PA *of that cardinality?*

We answer negatively in theorem 32, showing that in every infinite cardinality, there are spaces that continuously support some, but not all models of arithmetic in that cardinality. Thus, the underlying topology is able to distinguish the arithmetic structure. One might similarly ask whether it can distinguish the arithmetic *theory* of the structures.

Question 4. *If a topological space X continuously supports a model of* PA*, then does it continuously support a model of every completion of* PA*?*

This also we answer negatively in corollary 34, although the question remains open if one should insist that the topology is Hausdorff or that the space is uncountable.

Although we take PA as a central case, few of our arguments use the full strength of PA, and much of our analysis applies also to models of various weaker arithmetic theories. For example, some of the results apply to models of the theory PA^-, a very weak theory which drops the induction scheme completely from PA. The main theorem goes through for models of $\mathrm{I}\Delta_0 + \mathrm{Exp}$, which has induction for bounded-quantifier (Δ_0) assertions, plus the axiom Exp asserting the totality of the exponential function 2^x, a theory that suffices to develop the standard account of binary representations of numbers. Other results use a little more, such as the theory $\mathrm{I}\Sigma_1$, which includes instances of the induction scheme for existential (Σ_1) assertions—this is weaker than PA but stronger than $\mathrm{I}\Delta_0 + \mathrm{Exp}$. We refer the reader to [HP98] and [Kay91] for general background on arithmetical theories and their model theory.

2. Every countable model is continuously presented on the rationals

Let us prove the main theorem, which answers the main question.

Main Theorem 5. *Every countable model of* PA *has a continuous presentation on the rationals* $\mathbb{Q}$*.*

Proof. We shall prove the theorem first for the standard model of arithmetic $\langle \mathbb{N}, +, \cdot \rangle$. Every school child knows that when computing integer sums and products by the usual algorithms, the final digits of the result $x + y$ or $x \cdot y$ are completely determined by the corresponding final digits of the inputs x and y. Presented with only final segments of the input, the child can nevertheless proceed to compute the corresponding final segments of the sum or product.

$$
\begin{array}{rr}
& \cdots 1261 \\
+ & \cdots 153 \\
\hline
& \cdots 414
\end{array}
\qquad
\begin{array}{rr}
& \cdots 1261 \\
\times & \cdots 153 \\
\hline
& \cdots 3783 \\
& \cdots 6305 \\
& \cdots 1261 \\
& \cdots\cdots \\
\hline
& \cdots 933
\end{array}
$$

This phenomenon amounts exactly to the continuity of addition and multiplication with respect to what we call the *final-digits* topology on $\mathbb{N}$, which is the topology having basic open sets U_s, the set of numbers whose binary representations ends with the digits s, for any finite binary string s. (A similar idea can be used with any base, although the topologies can differ.) In the U_s notation, we include the number that would arise by deleting initial 0s from s; for

example, $6 \in U_{00110}$. Addition and multiplication are continuous in this topology, because if $x + y$ or $x \cdot y$ has final digits s, then by the school-child's observation, this is ensured by corresponding final digits in x and y, and so (x, y) has an open neighborhood in the final-digits product space, whose image under the sum or product, respectively, is contained in U_s.

Let us make several elementary observations about the topology. The sets U_s do indeed form the basis of a topology, because $U_s \cap U_t$ is empty, if s and t disagree on some digit (comparing from the right), or else it is either U_s or U_t, depending on which sequence is longer. The topology is Hausdorff, because different numbers are distinguished by sufficiently long segments of final digits. There are no isolated points, because every basic open set U_s has infinitely many elements. Every basic open set U_s is clopen, since the complement of U_s is the union of U_t, where t conflicts on some digit with s. The topology is actually the same as the metric topology generated by the 2-adic valuation, which assigns the distance between two numbers as 2^{-k}, when k is largest such that 2^k divides their difference; the set U_s is an open ball in this metric, centered at the number represented by s. (One can also see that it is metric by the Urysohn metrization theorem [Mun00, theorem 34.1], since it is a Hausdorff space with a countable clopen basis, and therefore regular.) By a theorem of Sierpiński [Sie20; Fra12], every countable metric space without isolated points is homeomorphic to the rational line $\mathbb{Q}$, and so we conclude that the final-digits topology on $\mathbb{N}$ is homeomorphic to $\mathbb{Q}$. We've therefore proved that the standard model of arithmetic $\mathbb{N}$ has a continuous presentation on $\mathbb{Q}$, as desired.

But let us belabor the argument somewhat, since we find it interesting to notice that the final-digits topology (or equivalently, the 2-adic metric topology on $\mathbb{N}$) is precisely the order topology of a certain definable order on $\mathbb{N}$, what we call the *final-digits* order, an endless dense linear order, which is therefore order-isomorphic and thus also homeomorphic to the rational line $\mathbb{Q}$, as desired. This will enable the generalization of corollary 7.
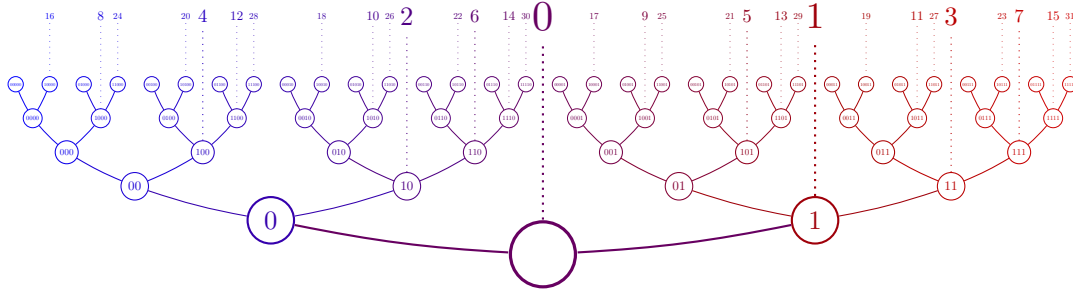

FIGURE 1. The final-digits order on the natural numbers

Specifically, the final-digits order on the natural numbers, pictured in figure 1, is the order induced from the lexical order on the finite binary representations, but considering the digits from right-to-left, giving higher priority in the lexical comparison to the low-value final digits of the number. To be precise, the final-digits order $n \triangleleft m$ holds, if at the first point of disagreement (from the right) in their binary representation, n has 0 and m has 1; or if there is no disagreement, because one of them is longer, then the longer number is lower, if the next digit is 0, and higher, if it is 1 (this is not the same as treating missing initial digits as zero). Thus, the even numbers appear as the left half of the order, since their final digit is 0, and

the odd numbers as the right half, since their final digit is 1, and 0 is directly in the middle; indeed, the highly even numbers, whose representations end with a lot of zeros, appear further and further to the left, while the highly odd numbers, which end with many ones, appear further and further to the right. If one does not allow initial 0s in the binary representation of numbers, then note that zero is represented in binary by the empty sequence. It is evident that the final-digits order is an endless dense linear order on $\mathbb{N}$, just as the corresponding lexical order on finite binary strings is an endless dense linear order.

The basic open set U_s of numbers having final digits s is an open set in this order, since any number ending with s is above a number with binary form $100\cdots0s$ and below a number with binary form $11\cdots1s$ in the final-digits order; so U_s is a union of intervals in the final-digits order. Conversely, every interval in the final-digits order is open in the final-digits topology, because if $n \lhd x \lhd m$, then this is determined by some final segment of the digits of x (appending initial 0s if necessary), and so there is some U_s containing x and contained in the interval between n and m. Thus, the final-digits topology is the precisely same as the order topology of the final-digits order, which is a definable endless dense linear order on $\mathbb{N}$. Since this order is isomorphic and hence homeomorphic to the rational line $\mathbb{Q}$, we conclude again that $\langle \mathbb{N}, +, \cdot \rangle$ admits a continuous presentation on $\mathbb{Q}$.

We now complete the proof by considering an arbitrary countable model M of PA. Let $\lhd^M$ be the final-digits order as defined inside M. Since the reasoning of the above paragraphs can be undertaken in PA, it follows that M can see that its addition and multiplication are continuous with respect to the order topology of its final-digits order. Since M is countable, the final-digits order of M makes it a countable endless dense linear order, which by Cantor's theorem is therefore order-isomorphic and hence homeomorphic to $\mathbb{Q}$. Thus, M has a continuous presentation on the rational line $\mathbb{Q}$, as desired.   □

The executive summary of the proof is: the arithmetic of the standard model $\mathbb{N}$ is continuous with respect to the final-digits topology, which is the same as the 2-adic metric topology on $\mathbb{N}$, and this is homeomorphic to the rational line, because it is the order topology of the final-digits order, a definable endless dense linear order; applied in a nonstandard model M, this observation means the arithmetic of M is continuous with respect to its rational line $\mathbb{Q}^M$, which for countable models is isomorphic to the actual rational line $\mathbb{Q}$, and so such an M is continuously presentable upon $\mathbb{Q}$.

The proof actually establishes the following more refined result.

Theorem 6. *Every countable model of* $\mathrm{I}\Delta_0 + Exp$ *has a continuous presentation on the rationals* $\mathbb{Q}$.

Proof. The proof of the main theorem does not use the full strength of PA, and can be carried out in the theory $\mathrm{I}\Delta_0 + \mathrm{Exp}$, which has sufficient strength for the handling of binary expansions of numbers that is at the core of the proof of theorem 5.   □

Let us briefly discuss a variant of the final-digits order, which some readers might find natural, but which generates a different topology than the final-digits topology. The order, pictured in figure 2, arises from the lexical order on infinitary binary sequences, if one embeds the finite binary sequences into that space by appending infinitely many zeros at the left. Thus, one imagines representing the number 6 with $\cdots 0000110$. This amounts to treating missing initial digits in the usual binary representations as 0. The order is manifestly different from the final-digits order; for example, 0 is now a least element. Nevertheless, this is a dense linear order with least element, and so it is order-isomorphic to $\mathbb{Q} \cap [0, 1)$, which is homeomorphic

FIGURE 2. A variant of the final-digits order on the natural numbers
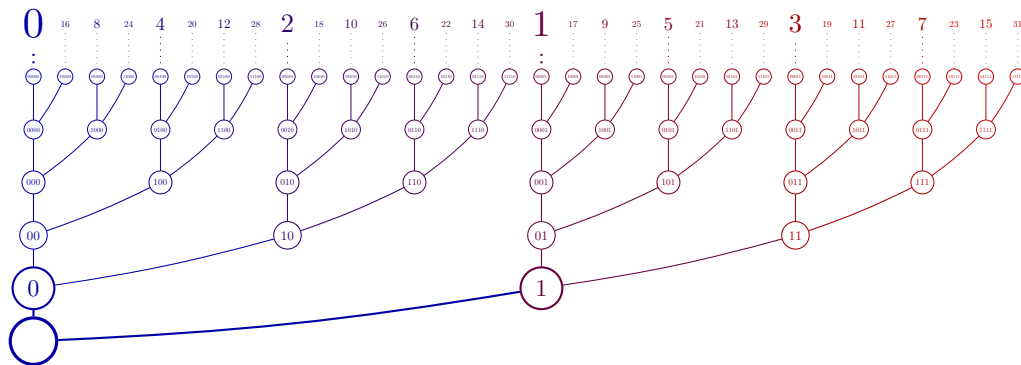
to $\mathbb{Q}$. We should like to point out, however, first, that this order does not give rise to the final-digits topology, since with this variant order, the set of numbers having final digits 110 has a least element, the number 6, and so it is not open in this order topology. Furthermore, second, the successor function $x \mapsto x + 1$ is not continuous with respect to this order, since $1 + 1 = 2$, and 2 has an open neighborhood containing only even numbers, but every open neighborhood of 1 contains both even and odd numbers, whose sums therefore will not all be in that neighborhood of 2. Thus, even the successor function is not continuous with respect to this modified order, and in this sense, it was not the right order to consider. Nevertheless, one can rescue things somewhat by noting that the final-digits topology is exactly the right-open topology of this order, with basic open sets consisting of half-open intervals $[n, m)$ in the order here, and therefore, by the argument of the main theorem, arithmetic is continuous with respect to the right-open topology of this order.

Corollary 7. *Every model of arithmetic $M \models \mathrm{PA}$ (actually $M \models \mathrm{I}\Sigma_1$ suffices) has a continuous presentation on the topological space $\mathbb{Q}^M$ with the order topology.*

Proof. The model M has a continuous presentation with respect to the final-digits order as defined in M. This order is seen as a countable endless dense linear order, from the perspective of M, and so by Cantor's theorem applied in M, which is provable in $\mathrm{I}\Sigma_1$, it is order-isomorphic and hence homeomorphic to the rationals $\mathbb{Q}^M$ of M. □

The investigation of saturated dense linear orders was initiated by Hausdorff (see [Hau02]), who introduced the familiar η notation for the countable endless dense linear order, and more generally η_α for the saturated endless dense linear order of cardality $\aleph_\alpha$, which exists precisely under the set-theoretical assumption that $\aleph_\alpha^{<\aleph_\alpha} = \aleph_\alpha$.

Corollary 8. *The order η_α, when it exists, supports a topological model of arithmetic.*

Proof. If η_α exists, then because $\aleph_\alpha^{<\aleph_\alpha} = \aleph_\alpha$, it follows that there are saturated structures of size $\aleph_\alpha$ in any first-order language of size less than $\aleph_\alpha$. Let M be such a saturated model of arithmetic of size $\aleph_\alpha$. Its rational numbers $\mathbb{Q}^M$ will therefore be a saturated endless dense linear order of size $\aleph_\alpha$, hence order-isomorphic to η_α. So η_α, when it exists, supports a topological model of arithmetic. □

Question 9. *Which topological spaces arise as $\mathbb{Q}^M$ for a model of arithmetic $M \models \mathrm{PA}$?*

The spaces $\mathbb{Q}^M$ are always at least a little homogeneous, since there are many order-isomorphisms of $\mathbb{Q}^M$ internal to M; for example, any finite partial order automorphism extends inside M to a full order automorphism. Also, $\mathbb{Q}^M$ is never complete as an order, since the model can see that $\sqrt{2}^M$ determines a cut in $\mathbb{Q}^M$ with no least upper bound; indeed, the order is totally disconnected.

Theorem 10. *If M is an uncountable model of arithmetic, then $\mathbb{Q}^M$ is not separable, and indeed, it has an uncountable discrete subset, and therefore admits an uncountable family of disjoint intervals.*

Proof. Every model of arithmetic M sits as a discrete set inside its rationals $\mathbb{Q}^M$. The model can place disjoint intervals around its integers, such as $(n - \frac{1}{3}, n + \frac{1}{3})$, and thereby produce an uncountable family of disjoint intervals. $\square$

In particular, $\mathbb{Q}^M$ is never a Suslin line, nor is it the space of irrational numbers. While theorem 10 rules out these latter spaces for $\mathbb{Q}^M$, it does not seem necessarily to rule them out as supporting topological models of arithmetic in some other way than as $\mathbb{Q}^M$.

We also know some things about definable suborders of models of arithmetic.

Theorem 11. *No nonstandard model of PA can define an infinite complete linear order.*

Proof. Suppose that $\langle L, \leq_L \rangle$ is a linear order definable in a nonstandard model of arithmetic $M \models$ PA. Since Ramsey's theorem is formalizable in PA, from the perspective of M there must either be an infinite increasing or decreasing sequence $\langle c_i \mid i \in M \rangle$. But the limit of c_k for standard k must exist, if L is complete, and from this M can define the standard cut, which is a contradiction. $\square$

Theorem 12. *If a linear order $\langle L, \leq_L \rangle$ is definable in a model $M \models$ PA, then every limit point in L has the same cofinality (from below or above) as the cofinality of M, and this is furthermore the same as the cofinality or coinitiality of L itself.*

Proof. If a point x is a limit point of L in M, then M thinks that x is the limit of an infinite sequence, and so x is the limit of a sequence of order type the same as the order type of M. Thus, the cofinality of x in L (from either direction) is the same as the cofinality of M itself. The same argument works for the cofinality and coinitiality of L itself, if the order does not have maximal or minimal elements, respectively. $\square$

Let us observe that the proof of the main theorem can be extended to the integers $\mathbb{Z}^M$ of the model of arithmetic M.

Theorem 13. *The ring of integers $\mathbb{Z}^M$ of any countable model M of PA, or indeed, of $I\Delta_0 + Exp$, admits a continuous presentation on the rational line $\mathbb{Q}$.*

Proof. Consider first the standard integers $\mathbb{Z}$. The analogue of the school-child's observation is that addition and multiplication of integers is continuous with respect to the signed-finals-digits topology, in the sense that if you know the final digits of the numbers and their sign (positive or negative), then you can know the corresponding final digits and sign of the output. If we represent integers by their binary digits with an indication for the sign—we represent zero by the empty sequence—then the basic open sets in this topology have the form U_{s+} or U_{s-}, for any nonempty binary sequence s, where the first set is the set of positive numbers whose binary expansion has final digits s, and the second set refers analogously to negative numbers, plus the sets of negative numbers U_-, positive numbers U_+ and the sets $U_{00\cdots0}$,

which are the numbers whose final digits end all in zeros, but including both positive and negative numbers. This topology is the same as the order topology of the signed-final-digits order, pictured in figure 3 below. This is an endless dense linear order, defined by the lexical order on the signed binary representation (but note how the sign affects the direction of the lexical comparison for negative numbers), and so the topology is homeomorphic to the rational line. By carrying out this argument inside any countable nonstandard model of arithmetic
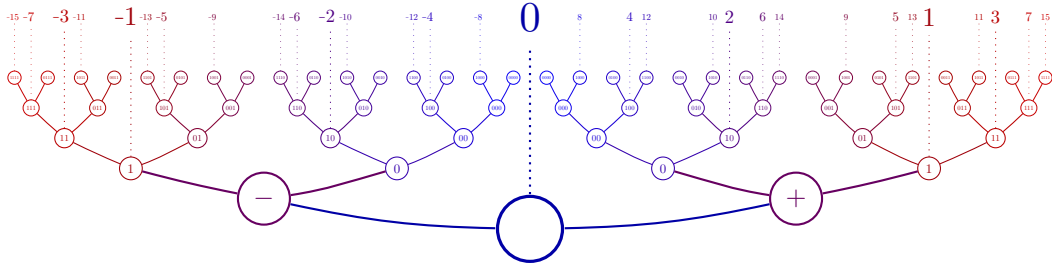

FIGURE 3. The signed-final-digits order on the integers

M, we conclude that $\mathbb{Z}^M$ also has a continuous presentation on $\mathbb{Q}^M$, which is order-isomorphic and hence homeomorphic to $\mathbb{Q}$. □

3. Some spaces do not support a topological model of arithmetic

Let us prove now that several common spaces do not support a topological model of arithmetic. For several familiar topological spaces, we shall prove this in a strong way, by proving that neither addition nor multiplication separately can be continuous, and indeed, that not even the unary sections of these functions can be continuous.

Theorem 14. *The Cantor space does not support a topological model of arithmetic. Indeed, no compact Hausdorff space supports a topological model of arithmetic.*

Proof. Suppose that M is a model of arithmetic (PA^- suffices), whose underlying set is a compact Hausdorff space. Let us show that addition is not continuous. For any $k \in M$, consider the interval $[k, \infty)$ in the sense of M. This interval is the image of M under the map $x \mapsto x + k$. If this map is continuous, then the image is compact and therefore also closed in M, and so the family of all such intervals is thus a nested family of nonempty closed sets with empty intersection, violating compactness. So addition is not continuous, not even separately in the first coordinate.

Let us also show that multiplication is not continuous. (In this part of the argument, since we use factorials, let us take 'arithmetic' to be the theory $\mathrm{I}\Delta_0 + \mathrm{Exp}$.) Consider the sets $M \cdot (k!) = \{\, n \cdot (k!) \mid n \in M \,\}$, where $k!$ means the factorial of k as seen by the model M. If multiplication is continuous, then these sets are the image of M under a continuous map and therefore are compact and hence closed. And they are nested, as k increases in M, with empty intersection. So we have a nested family of closed sets with empty intersection, contradicting compactness. So multiplication is not continuous, not even separately in the first coordinate. □

The proof establishes the stronger result, that if M is a compact Hausdorff space and $\langle M, +, \cdot \rangle$ is a model of arithmetic, then neither addition nor multiplication, separately, is continuous, and indeed, these functions are not even continuous separately in just one coordinate at a time.

Theorem 15. *The reals $\mathbb{R}$ do not support a topological model of arithmetic. Indeed, $\mathbb{R}$ does not support a model of arithmetic even whose successor function is continuous.*

Proof. Suppose that $\langle \mathbb{R}, \oplus, \otimes \rangle$ is a model of arithmetic. We need only that it is a model of successor arithmetic, a tiny part of PA^-, for we'll show that not even the successor function $x \mapsto x \oplus 1$ can be continuous, where by 1 here we mean the 1 of the model of arithmetic, which may not be the same as the real number 1. If the successor function were continuous, then the image of $\mathbb{R}$ under this map would be connected, since $\mathbb{R}$ is connected. But the range of the successor function omits exactly one point, zero, and the complement of a point in $\mathbb{R}$ is never connected. So the successor function cannot be continuous.   □

The argument generalizes to other one-dimensional spaces as follows.

Theorem 16. *The long line does not support a topological model of arithmetic. Indeed, no space that is connected, yet becomes disconnected upon the deletion of any point (or any two points), supports a topological model of arithmetic.*

Proof. The two-way version of the long line, which extends to ω_1 in both directions, is connected, yet becomes disconnected upon the deletion of any point. This would contradict the continuity of the successor function, who image omits one point. For the ordinary long line, that is, $[0, 1) \cdot \omega_1$, there is a least element, whose deletion does not disconnect the space, although the deletion of any two points will disconnect the space. This again violates continuity of the successor function, if we simply perform it twice, if necessary, for then the image will omit exactly two points.   □

Here is an alternative argument for the reals, which also applies to other spaces (but not to the long line).

Theorem 17. *The reals $\mathbb{R}$ do not support a topological model of arithmetic. Indeed, if a space is uncountable and connected, and has the further properties that (i) there is no descending ω_1-sequence of connected subspaces, and (ii) there is no partition of the space into uncountably many disjoint connected subspaces, then it does not support a topological model of arithmetic.*

Proof. Suppose that $\langle M, +, \cdot \rangle$ is an uncountable topological model of arithmetic, whose underlying space has the features mentioned in the theorem. Since we shall use factorials and the division algorithm, let us take 'arithmetic' here to refer to the theory $\mathrm{I}\Delta_0 + \mathrm{Exp}$. We split into two cases. First, suppose that the model happens to have an increasing ω_1-sequence of numbers $\langle k_\alpha \mid \alpha < \omega_1 \rangle$. (But see theorem 21 for why this case does not arise in $\mathbb{R}$ or in any Polish space.) In this case, we claim, multiplication cannot be continuous, even just in one coordinate. To see this, consider the image of the whole space $M \cdot (k_\alpha!)$ after multiplication by the factorial of k_α, as the model sees it. This is the continuous image of a connected set and therefore connected. Since these sets are descending as k_α increases, this would violate assumption (i) on the underlying space of M.

In the remaining case, since M is uncountable, there must be a number d in the model having at least ω_1 many predecessors $r < d$. For each such r, let A_r be the numbers with residue r modulo d as the model sees it. Thus, A_r is simply the image of M under the function

$q \mapsto dq + r$, which is a continuous function. So each A_r is connected, and we have therefore found a partition of M into uncountably many disjoint connected sets, which would violate assumption (ii) on the underlying space of M. □

The following observation will be useful in ruling out additional spaces.

Theorem 18. *If M is a topological model of arithmetic, then there is a continuous pairing function, that is, a continuous injection from M^2 to M. Indeed, there is a continuous bijection of M^2 with the even numbers of M.*

Proof. Every model of PA, and much weaker theories, has a polynomial pairing function. For example, the function $p(n, m) = \frac{1}{2}(n + m)(n + m + 1) + m$ is the Cantor pairing function, and this is provably a bijection between M^2 and M. We cannot generally divide by 2 in a continuous manner, however, and so to make a continuous injection, as opposed to bijection, we simply use $q(n, m) = (n+m)(n+m+1)+2m$, which is the double of the Cantor function. This is now a continuous injection of M^2 into M, and indeed, it is a continuous bijection of M^2 with the even numbers of M. □

Theorem 19. *The reals $\mathbb{R}^n$, in any finite dimension, do not support a topological model of arithmetic.*

Proof. Suppose that M is a topological model of arithmetic. By the pairing function argument explained in theorem 18, there is a continuous injective function $p : M^2 \to M$. But considerations of dimension theory in topology show that there can be no continuous injection of $\mathbb{R}^m$ into $\mathbb{R}^n$, when $n < m$. For example, if there were such a continuous injection, it would provide a homeomorphism of the m-dimensional cube $[0, 1]^m$ with its image in $\mathbb{R}^n$, which violates [Mun00, theorem 50.6], stating that closed subsets of $\mathbb{R}^n$ have dimension at most n. So M cannot have $\mathbb{R}^n$ as its underlying space. □

Question 20. *Does the Baire space support a topological model of arithmetic? Does $\mathbb{R}^\omega$ support a topological model of arithmetic? Does any uncountable Polish space support a topological model of arithmetic?*

The question is also motivated by a key result of [MMR91; MMR92], showing that every consistent first order theory in a countable language has a model whose universe is the Baire space and all of whose definable relations are F_σ as well as G_δ. So we know that there are low-complexity Borel models on the Baire space and other Polish spaces; the question is whether one can realize the model with continuous functions.[1]

The following observation partially addresses these questions.

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[1] Note that continuous functions in Polish spaces have closed graphs, or in other words, in the class F. In compact Hausdorff spaces, this is an equivalence, and so theorem 14 shows that in compact Hausdorff spaces, there is no model of arithmetic whose operations have a closed graph. Does this extend to $\mathbb{R}$ and the other spaces we have ruled out? Will Brian's observation [Bri18], posted on MathOverflow in answer to a question of the first author, may be relevant; he shows that there are no injections from $\mathbb{R}^m$ to $\mathbb{R}^n$ with a closed graph, when $m > n$; but note that the composition of functions with a closed graph need not necessarily have a closed graph.

Theorem 21. *If a model of arithmetic M is continuously presented on an uncountable Polish space, then there are no increasing or decreasing ω_1-sequences in the order. In particular, it is not κ-like for any cardinal κ.*

Proof. In this argument, we can take 'arithmetic' to mean PA^-, or indeed, Presburger arithmetic suffices. If addition $+$ is continuous, then the order $<$ is analytic, being the projection of a Borel relation via $a < b \longleftrightarrow \exists c\, a + c + 1 = b$. But a deep theorem due to Harrington and Shelah [HS82] shows that no analytic order can have uncountable increasing or decreasing chains. $\square$

4. The topology of a topological model of arithmetic

We should like to focus now on the underlying topology of a topological model of arithmetic, finding general features these models exhibit and investigating how one might change the topology to another related topology with respect to which the model remains continuous. In this section, let us take the phrase "model of arithmetic" to refer to the models of PA^-, which is much weaker than PA.

Theorem 22. *Every model of arithmetic M admits a continuous presentation with respect to the following topologies on M.*

  (1) *The discrete and indiscrete topologies on M.*
  (2) *The rational line $\mathbb{Q}^M$, as M sees it.*
  (3) *The initial-segment topology, whose open sets are initial segments of M.*
  (4) *The final-segment topology, whose open sets are final segments of M.*

Proof. Statement (1) is immediate, since all operations on a set are continuous with respect to discrete or indiscrete topologies. Statement (2) follow from the main theorem. For statement (3), consider the initial-segment topology. If $x + y \le k$, then for any $x' \le x$ and $y' \le y$, we would have $x' + y' \le k$. Thus, we have an open neighborhood of (x, y) whose image under addition is contained in the open set $[0, k]$, and so addition is continuous. A similar argument works for multiplication: if $xy \le k$, then any $x' \le x$ and $y' \le y$ will have $x'y' \le k$. So we have verified statement (3). For statement (4), we consider the final-segment topology. If $x + y \ge k$, then for any $x' \ge x$ and $y' \ge y$, we will have $x' + y' \ge k$, and so addition is continuous. Similarly, if $xy \ge k$, then for any $x' \ge x$ and $y' \ge y$, we have $x'y' \ge k$, and so multiplication is continuous, verifying statement (4). $\square$

The key for statements (3) and (4) was the monotone nature of addition and multiplication in a model of arithmetic; the argument would not work (and the result fails) in ordered rings, where multiplication by negative numbers is order reversing. Note that those topologies are not Hausdorff.

Theorem 23. *If M is a topological model of arithmetic, then M is upward homogeneous, meaning that whenever $a < b$ in M, then there is a continuous injection on M taking a to b. In particular, if b is isolated, then also all $a < b$ are isolated.*

Proof. Simply use the map $x \mapsto x + k$, where $k = b - a$ in M. $\square$

Topological models of arithmetic might fail to have the corresponding downward homogeneity, however, in light of the following.

Theorem 24. *Suppose that M is a topological model of arithmetic, and consider any element $b \in M$. Let M^* be the same model of arithmetic, but changing the topology to make every number $a \leq b$ isolated in M^*. Then M^* is continuous with respect to this new topology.*

Proof. The open sets of M^* have the form $U^* = U \cup A$, where U is open in M and $A \subseteq [0, b]$. To verify continuity in M^*, suppose that $x + y \in U^* = U \cup A$. If $x + y \leq b$, then both x and y also are $\leq b$, and so (x, y) is isolated in the plane, which is sufficient for this instance of continuity. Otherwise, we have $x + y \in U$, and so by the continuity of M there is a neighborhood of (x, y) in M^2 whose image under addition is contained in U. Thus, addition is continuous in M^*. A similar argument works for multiplication, with a slight wrinkle about zero. Namely, if $b \neq 0$ and $xy \leq b$, then both $x, y \leq b$ and we may proceed as with addition. And when $xy = 0$, then one of them must be 0, and in this case, $M \times \{0\}$ is an open set in the product space of M^* whose image maps to 0 under multiplication. So again we have verified continuity. □

Since isolating any set of points in a metric space preserves metrizability, it follows that M^* will be metrizable, if M was. One can view the construction of theorem 24 as an instance of the more general topology-blending constructions, which we now develop. These constructions are generally asymmetric, in that one is either adding open sets of small numbers or removing open sets of large numbers.

Theorem 25. *Suppose that M is a model of arithmetic that is continuous with respect to topologies τ_0 and τ_1, where τ_0 refines τ_1. For any initial segment $I \subseteq M$, let τ^* be the topology generated by sets of the form $U \cup A$, where $U \in \tau_1$ and A is an open subset of I in the subspace topology of τ_0. Then M is continuous with respect to the topology τ^*.*

Proof. If $x + y \in U \cup A$, then we break into cases. If $x + y \in U$, then we can find a neighborhood of (x, y) using the τ_1 topology. Otherwise $x + y \in A = W \cap I$ for some $W \in \tau_0$. In particular, $x + y \in I$ and so also $x, y \in I$. Thus, we can find a neighborhood of (x, y) using the τ_0 topology, and then restrict to the subspace I, which will give a neighborhood in the τ^* product topology. A similar argument works with xy, where again we need to pay attention to the case $xy = 0$ separately. □

Corollary 26. *If M is a topological model of arithmetic with topology τ, then for any initial segment $I \subseteq M$, the model M is also continuous with respect to the topology $\tau \restriction I$, whose open sets are of the form $U \cap I$, where U is open with respect to τ, plus M itself.*

Proof. The topology $\tau \restriction I$ is an instance of theorem 25, blending the indiscrete topology with the topology τ on I. □

Under the topology $\tau \restriction I$, the initial segment I appears the same as its subspace topology with respect to τ, but the points above I in the order of M become indiscrete. This method can be used even in the case $I = [0, b]$, where the initial segment has a maximal element.

Because of these topology-blending constructions, it follows that topological models of arithmetic need not be homogeneous. This makes the case of topological models of arithmetic very different from the related case of topological ordered rings, such as the integers $\mathbb{Z}^M$ of a model of arithmetic M, as these are always homogeneous: the translation maps $x \mapsto x + k$ are homeomorphisms mapping any point to any other desired point (the inverse map is continuous, since it is translation by $-k$). In particular, in such a topological ordered ring, if one point is isolated, then all points are isolated. It follows that if $\langle Z, +, \cdot \rangle$ is a countable ordered

ring, whose addition $+$ is continuous with respect to an underlying metric topology, then the underlying space is homeomorphic either to $\mathbb{N}$ or $\mathbb{Q}$. This is because if there is an isolated point, then all points are isolated and the topology is discrete; and otherwise, there are no isolated points, and by the theorem of Sierpiński we mentioned earlier, every countable metric space without isolated points is homeomorphic to $\mathbb{Q}$. With topological models of arithmetic, in contrast, one can achieve other countable metric spaces.

Theorem 27. *Every countable nonstandard model of arithmetic,* $\mathrm{I}\Delta_0 + Exp$, *has continuous presentations with respect to the countable metric spaces:*

  (1) The rationals $\mathbb{Q}$.
  (2) The rationals $\mathbb{Q}$, adjoined with finitely many isolated points.
  (3) The rationals $\mathbb{Q}$, adjoined with infinitely many isolated points.
  (4) The countable discrete metric space.

Proof. The main theorem gives $\mathbb{Q}$, and then by theorem 25 we can make finitely many or infinitely many points or all points isolated. $\qquad\square$

Note that the standard model has no continuous presentation on $\mathbb{Q}$ with infinitely many isolated points, since those points would have to be unbounded in $\mathbb{N}$ and thus every point would have to be isolated, by theorem 23.

The underlying topology-blending fact, of course, is the following elementary observation of topology.

Lemma 28. *If a binary operation on a set X is continuous with respect to topologies σ and also with respect to τ, then it is also continuous with respect to the topology generated by $\sigma \cup \tau$.*

Proof. Suppose that $\star$ is a binary operation on X, continuous with respect both to σ and τ separately, meaning that the corresponding product topology is used for X^2, the domain of the operation. The topology generated by $\sigma \cup \tau$ has basic open sets of the form $U \cap W$, where $U \in \sigma$ and $W \in \tau$. If $x \star y \in U \cap W$, then we can find open neighborhoods of (x,y) in the σ topology on X^2 and in the τ topology on X^2, which by intersection will provide an open neighborhood in the combined topology, whose image under $\star$ is contained in $U \cap W$. $\qquad\square$

For example, we may combine any given topology with the initial-segment topology, or with the final-segment topology, and produce a new topological model of arithmetic.

Theorem 29. *If M is a topological model of arithmetic, then so is M^*, which is the same model, but with a new topology, generated from the old by taking all initial segments of M also as open sets. And also M^{**}, which augments the topology of M with the final segments of M.*

Note that the final-digits topology on a model of arithmetic M, as in the main theorem, already makes all final-segments open, since for any n with binary representation s, the basic open set U_s consists of numbers all at least as large as n. So if one augments M with the initial segments to form M^*, then every point will become isolated.

Let us discuss the issue in the proof of theorem 18 about using the Cantor bijective pairing function

$$p(n,m) = \frac{1}{2}(n+m)(n+m+1) + m,$$

instead of its double. If dividing by 2 were continuous, then indeed we could use the bijective pairing function, and M^2 would be continuously bijective with M. But not every topological model of arithmetic has dividing-by-2 as a continuous function.

Theorem 30. *Every model of arithmetic M carries a topology in which dividing by 2 is not a continuous function on its even numbers, and in which also the predecessor function $x \mapsto x-1$ is not continuous on the positive numbers.*

Proof. Let M be any model of arithmetic. Consider this model under the topology $\tau \upharpoonright 17$, where τ is the discrete topology. So M has the discrete topology up to and including the number 17, but above 17 the topology is indiscrete. By corollary 26, this is a topological model of arithmetic.

If dividing by 2 were a continuous function on the even numbers of M, then since 30 gets mapped to 15, which is isolated, it would mean that 30 is isolated in the subspace topology of the even numbers. But the only open set containing 30 also contains all the numbers above 17, and so it isn't isolated in the even numbers. Contradiction.

A similar argument works with the predecessor function. Since $18 - 1 = 17$, if this function were continuous on the positive numbers, then 18 would have to be isolated in the subspace of positive numbers, but it isn't.   □

By theorem 27, the proof shows that the dividing-by-2 and predecessor functions can be discontinuous even when the underlying topology is a countable metric space. In these models, the map $x \mapsto 2x$ is not an open map, and neither is the successor function $x \mapsto x + 1$, since open maps would map isolated points to isolated points.

Question 31. *Is there a topological model of arithmetic M having no continuous bijection between M^2 and M?*

Note that subtraction and division are continuous with respect to the final-digits topology, in any model of arithmetic, by an analogue of the school-child's observation in the proof of the theorem 5.

Similar ideas as in theorem 30 provide a negative answer to question 3. Recall that a linearly ordered structure M is κ-*like*, if the cardinality of M is κ, but the cardinality of every proper initial segment of M is less than κ. It is well-known that every countable model of PA has a κ-like elementary extension for every infinite cardinal κ.

Theorem 32. *In every infinite cardinality κ, there is a topological space X of that size continuously supporting exactly the models of arithmetic of size κ that are not κ-like. Thus, X continuously supports some, but not all models of arithmetic of this size.*

Proof. Let X be a space consisting of the indiscrete topology on a set of size κ, augmented with κ many additional isolated points. If M is a model of arithmetic of size κ, then M is continuous with respect to the topology $\tau \upharpoonright b$, where τ is the discrete topology and $b \in M$. If M is not κ-like, then it has an element b having κ many predecessors, in which case $\tau \upharpoonright b$ has κ many isolated points and an indiscrete block of κ many additional points, making the space homeomorphic to X. But if the model M is κ-like, then it can have no continuous presentation on X, for in this case, the isolated points would have to be unbounded in M, and thus by theorem 23 it would follow that every point would have to be isolated, contrary to the nature of X. So X continuously supports exactly the non-κ-like models of arithmetic.   □

We can prove a weak dual to this result.

Theorem 33. *For every infinite cardinal κ, there is a topological space Y supporting a κ-like model, such that furthermore, only κ-like models are continuously presentable upon Y.*

Proof. Let M be any κ-like model, and let Y be the initial-segment topology of M. Suppose that N is continuously presented on this same topology. If N is not κ-like, then it has a point b with κ many predecessors $a < b$ in N. Let U be a neighborhood of b, which is not the whole space. It follows that U has size less than κ, and therefore it omits κ many $a < b$. For any such a, there is k_a such that $a + k_a = b$, and all such k_a are distinct. By continuity, there is an open neighborhood W of a such that $a' + k_a \in U$ for all $a' \in W$. Since the open sets of X are linearly ordered, $a \in W$ and $a \notin U$, it follows that $U \subseteq W$ and in particular, $b \in W$. Thus, $b + k_a \in U$ for all these a, which places κ many elements into U, contrary to assumption. $\square$

Thus, we have two topological spaces of size κ, namely the space X of theorem 32 and the space Y of theorem 33, each of which supports a topological model of arithmetic, but no model is continuously presented upon both of them. In this sense, the topology can make distinctions in the arithmetic structure. In the case $\kappa = \omega$, this provides a negative answer to question 4, since there is only one ω-like model of arithmetic: the standard model.

Corollary 34. *There is a countable topological space X, such that only the standard model of arithmetic N is continuously presented upon X, and furthermore this model has a unique such presentation. In particular, this answers question 4 negatively.*

Proof. Let X be the initial-segment topology on the standard model $\mathbb{N}$, which is continuously presented upon X by theorem 29. The standard model $\mathbb{N}$ is ω-like and it is up to isomorphism the only ω-like model of arithmetic. Since theorem 33 shows that only ω-like models have a continuous presentation upon X, it follows that only $\mathbb{N}$ admits such a presentation. In particular, all models presented upon X have the same theory (true arithmetic), which provides a strong negative answer to question 4.

Let us show that the presentation is unique. Suppose that $M = \langle \mathbb{N}, \oplus, \otimes \rangle$ is a model of arithmetic that is continuous with respect to the initial-segment topology on $\mathbb{N}$. Since $0^{\mathbb{N}}$ is the only isolated point, it follows from theorem 23 that $0^M = 0^{\mathbb{N}}$. Since $a \oplus 1^M = 1^{\mathbb{N}}$ for some a and $\{ 0, 1^{\mathbb{N}} \}$ is open, there must be an open set W such that $w \oplus 1^M \in \{ 0, 1^{\mathbb{N}} \}$ for all $w \in W$. But $w \oplus 1^M = 0$ is impossible, since $0^M = 0^{\mathbb{N}}$, and so $w \oplus 1^M = 1^{\mathbb{N}}$ for all $w \in W$. So W must have only one element, and so $W = \{ 0 \}$, the only open set with exactly one element. So $a = 0$ and therefore $1^M = 1^{\mathbb{N}}$. Continuing similarly by induction, one sees that M agrees completely with the standard model, and so the presentation is unique. $\square$

The arguments of theorem 33 and corollary 34 suggest that perhaps the initial segment topology can detect the order type of any model of arithmetic continuously presented upon it, and so we ask the following question.

Question 35. *If M is a model of arithmetic, and N is a model continuously presented with respect to the initial-segment topology of M, then must M and N have the same order $<$? Is this true for κ-like models M?*

The final argument of corollary 34 shows the answer is affirmative when $\kappa = \omega$. More generally, the same argument shows that if N is continuous with respect to the initial-segment topology of a model M, then the two models must have the same 0, the same 1 and so on through the standard part. The question is asking whether the pattern continues above this, so that the models have the same initial segments and hence the same order. Theorem 33 shows that when M is κ-like, then any N presented on the initial-segment topology of M will be κ-like, but it does not quite show for uncountable κ that the order of M and N agree or that these orders are isomorphic. Note that models of arithmetic can have the same order

without being isomorphic with respect to their addition and multiplication; for example, all countable nonstandard models of PA have the same order type: $\omega + \mathbb{Z} \cdot \mathbb{Q}$.

5. Topological models of arithmetic in the complex numbers

Let us conclude the paper by proving a connection between uncountable models of arithmetic and the complex numbers, which will provide many further instances of topological models of arithmetic.

Theorem 36. *Every model of arithmetic (and indeed, every model of* PA^- *) of size at most continuum has a presentation as a topological model, using a subspace of the real plane* $\mathbb{R}^2$. *Indeed, every such model is realized as a substructure of the complex numbers* $\langle \mathbb{C}, +, \cdot \rangle$.

Proof. The latter result is due to Josef Mlček [Mlč73]; see also the second author's blog post [Ham18]. Suppose that M is a model of PA^- of size at most continuum. Consider the integers of this model $\mathbb{Z}^M$, from which we may take the fraction field and then the algebraic closure, building a version of the field A of algebraic numbers over M.[2] So A is an algebraically closed field of characteristic zero, which has an elementary extension to such a field of size continuum. Since the theory of algebraically closed fields of characteristic zero is categorical in all uncountable powers, it follows that A is isomorphic to a substructure of the complex numbers $\mathbb{C}$, and isomorphic to $\mathbb{C}$ itself if M has size continuum. Since M is isomorphic to a substructure of $\mathbb{Z}^M$, which sits inside A, it follows that M is isomorphic to a substructure of $\mathbb{C}$, as claimed. Since complex number arithmetic is continuous with respect to the topology of the real plane, we thereby realize M as a topological model of arithmetic using this subset of $\mathbb{R}^2$. $\square$

In particular, every countable model of PA^- can be found as a substructure of the complex numbers. The same argument applied at higher cardinals shows that if k is the uncountable algebraically closed field of characteristic zero, then every model of arithmetic $M \models \mathrm{PA}^-$ of size at most the cardinality of k embeds into k.

Note that by theorem 21, if M is a κ-like model of arithmetic for some uncountable κ at most the continuum, then no subspace of $\mathbb{R}^2$ that continuously supports M is analytic.

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