RIEMANN ZETA VIA λ RINGS

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1. Introduction.

We would like to adopt the concept of Borger from [B] to study the Riemann ζ-function of integers. Borger claims, that the category of rings over F_1 should consist of λ-rings and the restriction of scalars from $\mathbf{Z}$ to F_1 takes any commutative ring R to its ring of Witt vectors $W(R)$ with its canonical λ structure. In this approach the mythical field F_1 is equal to the ring of integers $\mathbf{Z}$ with the canonical λ-structure:

$$\lambda^n(m) = \binom{m}{n}$$

We will denote it as F_1 in the rest of the paper. This definition does describe this object almost like a field. If λ-operations are part of the structure then the ideals in our rings should be preserved by them. It is easy to check that in F_1 there are no proper ideals preserved by λ-operations.

Recall that for any commutative ring R its ζ function ζ_R is defined via the following procedure. For an R-module X let $N(X)$ denote the cardinality of the set $End_R(X)$. We say that a module X is finite if $N(X)$ is finite. We denote by $P(R)$ the isomorphism classes of all non zero finite simple modules over R. Then we define:

Formula 1.1.

$$\zeta_R(s) = \prod_{X \in P(R)} (1 - N(X)^{-s})^{-1}$$

The above formula generalizes to any category with zero object, see [K]. If $R = \mathbf{Z}$ we recover the classical ζ-function of integers. For any R the simple module over R is equal to R/I where I is a maximal ideal in R. In our world of λ-rings we are going to take into account only λ-ideals in R. So for a λ-ring R its finite simple modules are the same as quotients R/I where I is a maximal λ-ideal in R. In such case the quotient R/I inherits a λ structure and this will be taken into account while calculating the numbers $N(X)$.

In Section 3 we classify all maximal λ-ideals in $W(\mathbf{Z})$. This leads to the observation that the categorical ζ-function of a λ-ring $W(\mathbf{Z})$ is the same as the classical ζ-function of $\mathbf{Z}$. In Section 4 we place the Riemann hypothesis within the framework of Weil conjectures. We prove that the classical ζ-function of $\mathbf{Z}$ can be calculated via counting the numbers of points of the affine line over F_1 with coefficients in correctly defined extensions of F_1.

2. Preliminaries on λ-rings.

Our rings are always commutative with units. Following [Y, Definition 1.10] we have:

Definition 2.1. *A λ-ring is a ring R together with functions*

$$\lambda^n : R \to R \quad (n \geq 0)$$

satisfying for any $x, y \in R$:
(1) $\lambda^0(x) = 1$,
(2) $\lambda^1(x) = x$,
(3) $\lambda^n(1) = 0$ for $n \geq 2$,
(4)$\lambda^n(x+y) = \sum_{i+j=n} \lambda^i(x)\lambda^j(y)$,
(5) $\lambda^n(xy) = P_n(\lambda^1(x), ..., \lambda^n(x); \lambda^1(y), ..., \lambda^n(y))$,
(6) $\lambda^n(\lambda^m(x)) = P_{n,m}(\lambda^1(x), ..., \lambda^{nm}(x))$.

Above P_n and $P_{n,m}$ are certain universal polynomials with integer coefficients obtained via symmetric functions theory (see [Y, Example 1.7 and 1.9]). By a homomorphism of λ-rings we mean a ring homomorphism which commutes with λ-operations. We say that $x \in R$ is of degree k, if k is the largest integer for which $\lambda^k(x) \neq 0$. If such finite k does not exist we say that x is of infinite degree. Observe that (by formula 4) the map

$$R \ni x \mapsto \sum_{i \geq 0} \lambda^i(x)t^i$$

is a homomorphism from the additive group of R to the multiplicative group of power series over R with constant term 1. We will denote this map as $\lambda_t(x)$. Observe also that $\lambda_t(0) = 1$ and hence $\lambda_t(-r) = \lambda_t(r)^{-1}$. Usually in the literature the set of power series over R with constant term 1, with addition defined by the power series multiplication and with properly defined multiplication is called the universal λ-ring of R and denoted $\Lambda(R)$ (see [Y, Chapter 2] for the full discussion on this concept). The universal Λ-ring of R can be defined for any commutative ring R but when R is a λ-ring then $\lambda_t : R \to \Lambda(R)$ is a λ-ring homomorphism. We will talk more about the universal construction in the next section.

Ring of integers $\mathbf{Z}$ carries the unique, canonical λ-ring structure described by the formula $\lambda^n(m) = \binom{m}{n}$. Similarly all integral monoid rings $\mathbf{Z}[\mathbf{M}]$ will be considered with the λ structure defined for any $m \in M$ by formulas

$$\lambda^1(m) = m$$

$$\lambda^i(m) = 0 \text{ for } i > 1.$$

We will always consider integral monoid rings with such λ-structure, because this structure is uniquely forced by the monoidal point of view on the field with one element (compare Proposition 2.3 below and the discussion which follows it). In most of our monoids the cancellation law will hold. It means that typically if $x, y, z \in M$ and $xy = xz$ then $y = z$. We have:

Lemma 2.2. *Let R be equal to the monoidal λ-ring $\mathbf{Z}[M]$ with the λ-structure defined above. Assume that cancellation law holds in M. Then in R only generators $m \in M \subset \mathbf{Z}[M]$ are of degree 1.*

Proof. We know that the elements of M are of degree 1. By separating positive and negative coefficients we get $\mathbf{Z}[M] \ni r = \Sigma_{i=1}^{k} a_i m_i - \Sigma_{j=1}^{l} b_j m_j$, where all a_is and b_js belong to $\mathbf{Z}$ and are greater than 0. Observe that the assumption that r is of degree 1 implies that $\lambda_t(r) = 1 + bt$ for a certain $b \in \mathbf{Z}[M]$. We can easily calculate λ_t-functions in the case of monoidal rings. Let $m \in M$ and a be a positive integer. Then

$$\lambda_t(am) = \lambda_t(m)^a = (1 + mt)^a$$

Hence we easily get

$$\lambda_t\left(\sum_{i=1}^{k} a_i m_i - \sum_{j=1}^{l} b_j m_j\right) = \prod_{i=1}^{k} (1 + m_i t)^{a_i} / \prod_{j=1}^{l} (1 + m_j t)^{b_j}$$

If r is of degree 1 we have equality

$$(*) \qquad \prod_{i=1}^{k} (1 + m_i t)^{a_i} = (1 + bt) \prod_{j=1}^{l} (1 + m_j t)^{b_j}$$

From this, by comparing coefficients at the highest degree of t we get

$$b = \prod_{i=1}^{k} (m_i)^{a_i} / \prod_{j=1}^{l} (m_j)^{b_j}$$

and hence $b \in M$. On the other hand, when we calculate the coefficient at the first degree in the equality **(*)** we get

$$\sum_{i=1}^{k} a_i m_i = b + \sum_{j=1}^{l} b_j m_j$$

But by the definition of a_is and b_js this is possible only when $r = b \in M$. □

Recall that M_{ab} denotes the category of commutative monoids with unit. Let $Ring^{\lambda}$ stand for the category of commutative unital rings with λ-structure. We have:

Proposition 2.3. *The functor $M_{ab} \to Ring^{\lambda}$, which takes a monoid M to the λ-ring $\mathbf{Z}[M]$ with the λ-structure defined above has a right adjoint $Ring^{\lambda} \to M_{ab}$ which takes a λ-ring R to the multiplicative monoid R_1 of its elements of degree not exceeding* 1*.*

Proof. By [Y, Proposition 1.13] we know that in any λ-ring the product of 1-dimensional elements is again 1-dimensional (or equal to 0). Hence R_1 is a well defined multiplicative submonoid of R considered here as the multiplicative monoid. If $f \in Mor_{Ring^{\lambda}}(R, S)$ then f carries 1 dimensional elements of R to 1 dimensional elements of S or to 0 by the definition of a λ-homomorphism. Hence our right adjoint is well defined. The rest of the proof is obvious.

The λ-operations on a ring R define on it the sequence of Adams operations $\psi^k : R \to R$ which are natural ring homomorphisms. They can be defined by the Newton formula:

$$\psi^k(x) - \lambda^1(x)\psi^{k-1}(x) + ... + (-1)^{k-1}\lambda^{k-1}(x)\psi^1(x) = (-1)^{k-1}k\lambda^k(x)$$

For their properties see [Y, chapter 3]. It is straightforward to check that the canonical λ-structure on $\mathbf{Z}$ defines trivial Adams operations ($\psi^k = id$ for any k) and the formula $m \mapsto m^k$ for $m \in M$ determines the kth Adams operation on the monoidal ring $\mathbf{Z}[\mathbf{M}]$. Adams operations can be viewed always as an action on a considered structure by the multiplicative monoid $\mathbf{N}^*$ of natural numbers. Every object M of M_{ab} has naturally one such a structure given by identifying $k \in \mathbf{N}^*$ with $\psi^k : M \to M$ where $\psi^k(m) = m^k$. This structure is obvious in M_{ab} and adds very little while studying monoids, but should be reflected always, when we want to induce structures from M_{ab} to other (abelian) categories. This action will be addressed as an action of Adams operations on M. Proposition 2.3 can be viewed as a statement about adjoint functors between categories with objects carrying the action of $\mathbf{N}^*$. It is easy to check that the $\mathbf{N}^*$ action on $\mathbf{Z}[M]$ given by $k(m) = m^k$ while treated as the action of Adams operations forces to have λ-structure on $\mathbf{Z}[M]$ satisfying $\lambda^i(m) = 0$ for $i > 1$.

Recall that $\mathbf{N}^+$ denotes the monoid of natural numbers with addition. Using it we can define for any ring R the polynomial ring over R via the formula

$$R[x] = R[\mathbf{N}^+] = \mathbf{R} \otimes \mathbf{Z}[\mathbf{N}^+].$$

We will consider $\mathbf{Z}[\mathbf{N}^+]$ as polynomial ring over F_1 in the rest of the paper, with λ-structure defined like for any other monoidal ring. Moreover, for any λ-ring R we have well defined λ-structure on $R[x]$ because tensor product of rings inherits it from the λ-structures of the factors.

Remark 2.4. As we said before the λ-structure on $\mathbf{Z}$ is unique. This is not the case with monoidal rings $\mathbf{Z}[\mathbf{M}]$. We want to view the results of [Be] and our Proposition 2.3 as strong indication, that the restriction of considered λ

structures on our monoidal rings to the ones defined by Adams operations on monoids is well justified. This restriction is crucial for the whole of the paper and more generally for the whole of our approach to studying the Riemann ζ-function of integers.

Definition 2.5. *Let R be a λ-ring and I is an ideal in R. We will call it a λ-ideal if it is preserved under the action of λ^k, for any $k > 0$.*

It is straightforward to check that if we divide a λ-ring by a λ-ideal then R/I carries the induced λ-structure and the quotient homomorphism $R \to R/I$ is a homomorphism of λ-rings. Of course the opposite is also true: a kernel of the λ-rings homomorphism is a λ-ideal. It is important for computations that an ideal I in a λ-ring R with $\mathbf{Z}$-torsion free quotient is a λ-ideal if and only if it is preserved by the Adams operations (see [Y, Corollary 3.16]).

3. Categorical zeta function for Witt vectors

We start from recalling the definition of the universal λ-ring of a commutative ring R.

Definition 3.1. *For any ring R we define its universal λ-ring $\Lambda(R)$ in the following way:*

- *As a set $\Lambda(R)$ is equal to the set of formal power series over R with leading term equal to 1.*
- *Ring operations $+_\lambda$ and $\times_\lambda$ are defined via the formulas*

$$(1+\sum_{i=1}^{\infty} a_i t^i) +_\lambda (1+\sum_{i=1}^{\infty} b_i t^i) \;=\; (1+\sum_{i=1}^{\infty} a_i t^i)(1+\sum_{i=1}^{\infty} b_i t^i)$$

$$(1+\sum_{i=1}^{\infty} a_i t^i) \times_\lambda (1+\sum_{i=1}^{\infty} b_i t^i) \;=\; (1+\sum_{i=1}^{\infty} P_i(a_1,...,a_i;b_1,...,b_i)t^i$$

- *For $x = 1+\Sigma_{i=1}^{\infty} a_i t^i$ the λ-operations are defined via the formulas:*

$$\lambda^m(x) = \sum_{i=1}^{\infty} P_{i,m}(a_1,...,a_{im})t^i.$$

It is proved in [Y, Theorems 2.5 and 2.6] that the above structure defines a λ-ring. As we said before, the ring of integers $\mathbf{Z}$ carries the unique, canonical λ-ring structure described by the formula $\lambda^n(m) = \binom{m}{n}$. Clearly $\mathbf{Z}$ has no proper λ-ideals because $\binom{n}{n} = 1$. When I is a λ-ideal in R then the quotient ring carries the natural λ-ring structure. Moreover when $\varphi : R \to S$ is a λ-ring homomorphism then $ker(\varphi)$ is a λ-ideal.

Lemma 3.2. *Let* $I_k(R) = \{1 + \Sigma_{i=1}^{\infty} a_i t^i \mid a_1 = ... = a_{k-1} = 0\}$*. Then* $I_k(R)$ *is an ideal in* $\Lambda(R)$*.*

Proof. It is obvious that $I_k(R)$ is preserved by $+_\lambda$. It follows from the definition of the polynomials P_i that $P_i(a_1, ..., a_i; b_1, ..., b_i)$ contains only terms which are multiples of some a_p and some b_q for $1 \leq p, q \leq i$.

Lemma 3.3. *Assume that* I *is a proper* λ*-ideal in* $\Lambda(R)$*. Then* I *contains an element* $1 + \Sigma_{i=1}^{\infty} a_i t^i$ *with* $a_1 \neq 0$*.*

Proof. See calculations in [Y, Example 1.7]. It follows that

$$P_{1,m}(a_1, ..., a_m) = a_m$$

Hence for $x = 1 + \Sigma_{i=1}^{\infty} a_i t^i$ with $a_m \neq 0$ we have $\lambda^m(x) = 1 + \Sigma_{i=1}^{\infty} b_i t^i$ with $b_1 = a_m$.

Lemma 3.4. *For a finite field* F_p *its universal* λ*-ring* $\Lambda(F_p)$ *contains no proper* λ*-ideals.*

Proof. The universal λ-ring $\Lambda(F_p)$ is isomorphic to a ring of p-adic integers $\hat{Z}_p$. To see this observe that $\Lambda(F_p)$ is isomorphic to a ring of Witt vectors $W(F_p)$ via the Artin-Hasse exponential, see [Y, Theorem 4.16]. For the isomorphism between $W(F_p)$ and $\hat{Z}_p$ see [E, Chapter I.5].

The ring $\hat{Z}_p$ is a principal ideal domain witch is local with the maximal ideal generated by $p \cdot 1$. Hence the same is true about $\Lambda(F_p)$. The unit in the latter ring is equal to $1 + t$ so the generator of the maximal ideal is equal to $1 + t^p$. Combining this with lemmas 3.2 and 3.3 gives us the proof. □

Now we want to understand maximal λ-ideals in $\Lambda(\mathbf{Z})$. For a given prime number p let $I^p \subset \Lambda(\mathbf{Z})$ consists of these power series which have all coefficients divisible by p. It follows directly from the definitions that I^p is an λ-ideal in $\Lambda(\mathbf{Z})$. It follows from lemma 3.1 that it is the maximal λ-ideal because the quotient $\Lambda(\mathbf{Z})/\mathbf{I^p}$ is isomorphic to $\Lambda(F_p)$.

Theorem 3.5. *Let* I *be a maximal* λ*-ideal in* $\Lambda(\mathbf{Z})$*. Then* $I = I^p$ *for a certain prime number* p*.*

Proof. Assume that I is a maximal λ-ideal in $\Lambda(\mathbf{Z})$ which is different than I^p for any prime number p. Then for any p the image of I under the quotient homomorphism $\Lambda(\mathbf{Z}) \to \mathbf{\Lambda(F_p)}$ has to be the whole $\Lambda(F_p)$. We want to show that $I = \Lambda(\mathbf{Z})$.

Let $W_k = \Lambda(\mathbf{Z})/\mathbf{I_k}(\mathbf{Z})$. We have the equality $\Lambda(Z) = \lim_k W_k$ where the structure map

$$W_k \to W_{k-1}$$

is defined as a quotient map where we divide W_k by the image of $I_{k-1}(\mathbf{Z})$. Let $V_k = I/I \cap I_k(\mathbf{Z})$. Then $I = lim_k V_k$. The embedding $\iota : I \hookrightarrow \Lambda(\mathbf{Z})$ is a map of inverse limits and it restricts to maps $\iota_k : V_k \to W_k$ which commute with the structure maps. It is enough for our purposes to show that these maps are isomorphisms. By construction we know that these maps are injective.

The ideal I is a λ-ideal. Lemma 2.5 tells us that I maps onto $\Lambda(F_p)/I_2(F_p) = F_p$ for any prime number p. This implies that I maps onto $\Lambda(\mathbf{Z})/\mathbf{I_2}(\mathbf{Z}) = \mathbf{Z}$ as well. Now we can finish the proof by induction. First of all observe that for any k both W_k and V_k are additively the free abelian groups of rank $k-1$. This is easily follows from induction and the short exact sequences:

$$0 \to Z \to W_k \to W_{k-1} \to 0$$

$$0 \to Z \to V_k \to V_{k-1} \to 0$$

Assume that ι_k is an isomorphism for $k < n$. For any prime number p we have a commuting diagram:

$$\begin{array}{ccc} V_{n-1} & \leftarrow & V_n \\ \downarrow & & \downarrow \\ W_{n-1} & \leftarrow & W_n \\ \downarrow & & \downarrow \\ W_{n-1}(F_p) & \leftarrow & W_{n-1}(F_p) \end{array}$$

which on the level of abelian groups shows up as follows:

$$\begin{array}{ccc} Z^{n-2} & \leftarrow & Z^{n-1} \\ \downarrow & & \downarrow \\ Z^{n-2} & \leftarrow & Z^{n-1} \\ \downarrow & & \downarrow \\ F_p^{n-2} & \leftarrow & F_p^{n-1} \end{array}$$

The right upper vertical map is injective. We know that both compositions of vertical maps are epimorphisms. This holds for any prime number p so the cokernel of the right upper vertical map has to be trivial. □

The ring of integers with scalars restricted from $\mathbf{Z}$ to F_1 is equal to the ring of Witt vectors $W(\mathbf{Z})$. As a model of $W(\mathbf{Z})$ we will use $\Lambda(\mathbf{Z})$. Classically the ring of Witt vectors was defined via Witt polynomials. Theorem [Y, 4.16] tells us that both descriptions give us isomorphic λ-rings with an isomorphism $E_R : \Lambda(R) \to W(R)$ given by the so-called Artin-Hasse exponential. We know what are the maximal λ-ideals in $\Lambda(\mathbf{Z})$ and we know the quotients $\Lambda(\mathbf{Z})/\mathbf{I_p} = \mathbf{\Lambda(F_p)}$. In order to calculate the categorical ζ function of $\mathbf{Z}$ we have to compute the order of $N(\Lambda(F_p))$.

Proposition 3.6. *For any prime p the order of $N(\Lambda(F_p))$ is equal to p.*

Proof. We want to count only these homomorphism $(\Lambda(F_p) \to (\Lambda(F_p)$ which preserve λ-structure and are module homomorphisms over $\Lambda(\mathbf{Z})$. Hence they are fully determined by their image of the unit of $\Lambda(F_p)$ equal to $1+t$. The image of an element of degree 1 must be of degree less or equal to 1. There are no other restrictions so we get p different homomorphisms $f_i : (\Lambda(F_p) \to (\Lambda(F_p)$ defined by the formula $f_i(1+t) = 1+it$, where $i = 0, 1, ..., p-1$.

Corollary 3.7. *The categorical ζ-function of the λ-ring $W(\mathbf{Z})$ is equal to the classical ζ-function of $\mathbf{Z}$.*

4. Zeta of integers via F_1[x].

As was written in the Introduction, we can perform the calculation of the Riemann ζ-function for any commutative ring R using the category of R-modules. The finite simple objects in the category of R-modules correspond to the maximal ideals $I \subset R$ with finite quotient. For any ring R let $Rings/R$ denote the category of rings over R. One checks immediately that the cardinality of $Hom_{Rings/R}(R[x], R \triangleright R/I)$ is the same as the cardinality of $Hom_{R-mod}(R/I, R/I)$. Hence instead of calculating cardinality of the set of endomorphisms we can calculate the number of $R \triangleright R/I$ points of the affine line over R .

There are good analogues of the category of modules over an object X of an abstract category $\mathcal{C}$ which has 0 and all finite limits. Beck in [Bec] defined them as abelian group objects in the category of objects over X (see also [H, chapter 2]). It is shown in [H] that the category of abelian group objects in the category of rings over a given ring R is equivalent to the category of R-modules, where an R-module X defines the square zero extension of R with X as a square-zero ideal. In the case of $R = \mathbf{Z}$ we get, as expected, the category of abelian groups. An abelian group X corresponds to the square zero extension $\mathbf{Z} \triangleright X$. The finite simple abelian group objects in the category of rings over $\mathbf{Z}$ are easily seen to come from the simple abelian groups (finite cyclic groups C_p of prime order p). For a given p we see that $N(C_p)$ is equal to the cardinality of the set $Hom_{Rings/\mathbf{Z}}(\mathbf{Z}[x], \mathbf{Z} \triangleright C_p)$. The polynomial ring $\mathbf{Z}[x] = \mathbf{Z}[N^+]$ is treated as a ring over $\mathbf{Z}$ via the map which takes x to 0. This is not our choice but it is forced by our monoidal approach and the fact that the one point unital monoid $\mathbf{1}$ is a zero object in the category M_{ab}. All this means that we have the geometrical method for calculating the categorical ζ of integers. We just have to count the $\mathbf{Z} \triangleright C_p$-points of the affine line over $\mathbf{Z}$.

Observe that we can perform the same calculations in the category M_{ab}, where the role of integers is played by the field of one element in the sense of [D]. But this gives us no new insight because in the world of monoids the field of one element is represented by one point monoid $\mathbf{1}$ consisting of 1 only, so we have equality of categories $M_{ab}/\mathbf{1} = M_{ab}$. But of course, in the spirit of our previous statements, the calculation from [Be] can be presented as counting the $\mathbf{1} \triangleright M$ points of the affine line $\mathbf{N}^+$ over $\mathbf{1}$.

Below we show that we get the Riemann ζ-function of the integers via counting the number of $F_1 \triangleright M$-points of $F_1[x]$ in the category of rings over F_1 where M runs through finite simple objects in the category of F_1-modules. We have to start from describing the latter category in some accessible way. We will follow closely [H] because in [H, chapter 2] this is done in full details for any λ-ring. The constructions uses the functor W from unital commutative rings to λ-rings which takes any ring R to its ring of Witt vectors $W(R)$. Originally the functor W was defined for rings with multiplicative unit. But the universal polynomials which define addition, multiplication and opposite in $W(R)$ do not use multiplicative unit so using the same formulas one can define the value of W on non-unital rings.

Recall that if R is a λ-ring then it comes with the λ-ring map $\lambda_R : R \to W(R)$ which is defined by lambda operations on R. As was explained in 3.4, if $\Lambda(R)$ denote the ring of invertible formal power series over R then $\lambda_R = E \circ \lambda_t$ where E is the Artin-Hasse exponential isomorphism of $\Lambda(R)$ and $W(R)$ (see [Y, chapter 4]) and

$$\lambda_t(r) = \sum_{i=0}^{\infty} \lambda^i(r)t^i$$

As it is proved in [H], the category of modules over a λ-ring R, which is equal to the category $(Ring^\lambda/R)^+$ of abelian group objects in $Ring^\lambda/R$, is equivalent to the category $R-mod^\lambda$ of λ-modules over R. A λ-module over R is an R module M with a map $\lambda_M : M \to W(M)$ which is equivariant with respect to the λ-structure of R. Here $W(M)$ denotes the Witt ring construction applied to the non-unital ring M with trivial multiplication. It is easy to check that in this case $W(M)$ has also trivial multiplication and additively is equal to the infinite product of M. It is shown in [H, Lemma 2.2] that we have an isomorphism of rings

$$i : W(R) \triangleright W(M) \to W(R \triangleright M)$$

which is induced by the canonical inclusions of R and M into $R \triangleright M$. A λ-module M corresponds in the equivalence of $(Ring^\lambda/R)^+$ and $R - mod^\lambda$ to the λ-ring $R \triangleright M$ with the λ-ring structure defined by the composition

$$R \triangleright M \xrightarrow{\lambda_R \oplus \lambda_M} W(R) \triangleright W(M) \xrightarrow{i} W(R \triangleright M)$$

We have another description of the category $R - mod^\lambda$ (see [H, Remark 2.6]). If M is an object of this category and $\lambda_M : M \to W(M)$ is a structural map then it has components $\lambda_{M,n} : M \to M$ because as sets $W(M) = \prod_N M$. Easy calculation shows that $\lambda_{M,n}$ is $\psi_{R,n}$ equivariant, where $\psi_{R,n}$ is the nth Adams operation of R. This gives us description of the category $R-mod^\lambda$ as a category of left modules over a twisted monoid algebra $R^\psi[\mathbf{N}^*]$ where the multiplicative monoid $\mathbf{N}^*$ acts on any object M through the maps $\lambda_{M,n}$.

With the understanding of the category $R-mod^{\lambda}$ presented above we can come back to our situation and analyse the category F_1-mod^{λ}. Observe that the Newton formula which relates Adams and λ-operations gives $\psi_{F_1,n} = id$ for any natural n. This implies that $\lambda_{M,1} = id$ and for $n > 1$, $\lambda_{M,n} : M \to M$ is any (additive) group homomorphism.

Lemma 4.1. *Every object (M, λ_M) in $F_1 - mod^{\lambda}$ consists of an abelian group M and a sequence of group homomorphisms $\lambda_{M,n} : M \to M$ satisfying $\lambda_{M,n} \circ \lambda_{M,m} = \lambda_{M,mn}$ and $\lambda_{M,1} = id$. Morphisms $(M, \lambda_M) \to (P, \lambda_P)$ are given as group homomorphisms $f : M \to P$ which satisfy $f \circ \lambda_{M,n} = \lambda_{P,n} \circ f$ for any natural n.*

The description of $F_1 - mod^{\lambda}$ was achieved before the statement of the lemma. But let us make here one comment. Our category of F_1-modules is the same as the category of modules over $\mathbf{Z}[\mathbf{N}^*]$ which is almost the same as the monoid algebra over $\mathbf{Z}$ of the multiplicative monoid of integers. Hence we did not leave the old approach to the field of one element, presented for example in [D], but it seems that we are getting more subtle methods of approaching the Riemann ζ-function.

Lemma 4.2. *Assume M is a finite simple object in $F_1 - mod^{\lambda}$. If the set*

$$Hom_{Rings/F_1}(\mathbf{Z}[x], \mathbf{Z} \triangleright M)$$

has finite cardinality different from 0 then M is of the form (C_p, λ_{C_p}), where C_p is the cyclic group of prime order p and $\lambda_{C_p,n} = 0$ for $n \geq 1$.

Proof. Assume

$$1 < n(\mathbf{Z} \triangleright M) = |Hom_{Rings/F_1}(\mathbf{Z}[x], \mathbf{Z} \triangleright M)| < \infty.$$

Assume that $\varphi \in Hom_{Rings/F_1}(\mathbf{Z}[x], \mathbf{Z} \triangleright M)$. The structure of the semi-direct product implies that $\varphi(x) = (0, m)$ for a certain $m \in M$. Because φ is a λ-ring homomorphism it has to commute with λ-operations on the source and the target. Recall that in $\mathbf{Z}[x]$, $\lambda^n(x) = 0$ for $n > 1$. Hence for $n > 1$ we calculate

$$0 = \varphi(\lambda^n(x)) = \lambda^n(\varphi(x)) = \lambda^n((0, m)).$$

It means that we are looking for such objects (M, λ_M) and $m \in M$ which give us vanishing of higher λ-operations on elements $(0, m) \in \mathbf{Z} \triangleright M$. Now we can use the general formula for the Artin-Hasse isomorphism [Y, Section 4.2]. For any ring R if $f(t) = 1 + \Sigma a_i t^i \in \Lambda(R)$ then we write $f(t) = \prod(1 - (-1)^i b_i t^i)$ and the Artin-Hasse isomorphism $E : \Lambda(R) \to W(R)$ takes f to the sequence $(b_1, b_2, b_3, ...)$. This implies immediately that if λ-operations act trivially on $r \in R$ then this element has trivial (above the first) Witt coordinates in $W(R)$.

Now we come back to our considerations. The map $\lambda_M : M \to W(M)$ takes $m \in M$ to $(\lambda_{M,1}(m), \lambda_{M,2}(m)), \lambda_{M,3}(m), ...)$. Assume that

$$\lambda_{\mathbf{Z} \triangleright M}((0, m)) = ((a_1, y_1), (a_2, y_2), ...).$$

It is obvious that $a_n = 0$ for any positive n. Artin-Hasse isomorphism described above tells us that $y_1 = m$ and $y_n = 0$ for $n > 1$. On the other hand calculations from Addendum 2.3 of [H] tell us that

$$\lambda_{M,n}(m) = y_n = 0.$$

Hence $\lambda_{M,n}(m) = 0$ as we wanted to show.

Our assumption is that M is a finite simple object in $F_1 - mod^\lambda$. The λ-operations act trivially on M so M has to be simple as an abelian group. This observation implies our lemma.

Corollary 4.3. *Recall that $\zeta_{\mathbf{Z}}$ denotes the Riemann ζ-function of integers. Let $\mathcal{C} = F_1 - mod^\lambda$ and $n(M) = |Hom_{Rings/F_1}(\mathbf{Z}[x], \mathbf{Z} \triangleright M)|$. We have*

$$\zeta_{\mathbf{Z}}(s) = \prod_{M \in P'(\mathcal{C})} (1 - n(M)^{-s})^{-1}$$

where $P'(\mathcal{C})$ denote those classes M from $P(\mathcal{C})$ for which $n(M) \neq 1$.

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