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ON EXT AND HYPEREXT GROUPS IN THE CATEGORY OF FUNCTORS

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Abstract. Let R be a commutative noetherian ring with unit and let $\mathcal{F}$ denote the category of functors from the category of finitely generated R-modules to R-modules. Let $I \in \mathcal{F}$ denote the inclusion functor. We study homological algebra of I in the category $\mathcal{F}$ (Ext-groups) and its generalization when we allow coefficients to be chain complexes in $\mathcal{F}$ (Hyperext-groups). We compare the Ext-groups of I with coefficients in arbitrary $F \in \mathcal{F}$ with Ext-groups of I with coefficients in stable derived functors of F. The latter groups are relatively easily calculable because the stable derived functors are linear. On the other hand, known calculations of Ext-groups of I with coefficients in F can shed light on the stable derived functors of F which are hard to approach.

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1. Introduction

Let R be a commutative noetherian ring with unit and let $\mathcal{F}$ denote the category of functors from the category of finitely generated R-modules to R-modules. The category $\mathcal{F}$ is abelian with enough projective and injective objects so we can talk about homological algebra in $\mathcal{F}$. For a given $F \in \mathcal{F}$ let $L^s_i F$ ($R^s_i F$) denote *ith* left (right) stable derived functor of F in the sense of Dold and Puppe (see [3] or [11], Chapter 6). Let $I \in \mathcal{F}$ denote the inclusion functor. In section 3 we establish some relations between $Ext^*_{\mathcal{F}}(F, I)$ and $\oplus_i Ext^*_{\mathcal{F}}(L^s_i F, I)$ where both sides are considered as right $Ext^*_{\mathcal{F}}(I, I)$-modules. Similarly, for a special R, we get a relation between left $Ext^*_{\mathcal{F}}(I, I)$-modules $Ext^*_{\mathcal{F}}(I, F)$ and $\oplus_i Ext^*_{\mathcal{F}}(I, R^s_i F)$. Section 4 is devoted to studying the dual situation of $Ext^*_{\mathcal{F}}(I, F)$ and $\oplus_i Ext^*_{\mathcal{F}}(I, L^s_i F)$. The stable derived functors are additive and our results show that we can treat them as additive coefficients needed for calculating $Ext^*_{\mathcal{F}}(I, F)$ and $Ext^*_{\mathcal{F}}(F, I)$. Note that in *calculable* cases (for example when R is a finite field) the $Ext^*_{\mathcal{F}}(I, I)$-modules $Ext^*_{\mathcal{F}}(I, F)$ and $Ext^*_{\mathcal{F}}(F, I)$ are fully understood for any additive F.

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The possibility of reducing an arbitrary functor to its stable satellites in order to perform $Ext_{\mathcal{F}}$-calculations came as an effect of studying Mac Lane's Q-construction QF related to F, defined and developed in [[5], Sections 6 and 7]. By definition QF is a nonnegative chain complex in $\mathcal{F}$. The homology functors of QF are isomorphic to the left stable derived functors of F. We have $(QF)_0 = F$ and hence QF comes with the map of chain complexes $F \to QF$ where, as usual, F is treated as a chain complex concentrated in dimension 0. This map induces for any i a homomorphism of hyperext groups

$$\mathbf{Ext}^i_{\mathcal{F}}(QF, I) \to Ext^i_{\mathcal{F}}(F, I) \tag{1.1}$$

$$Ext^i_{\mathcal{F}}(I, F) \to \mathbf{Ext}^i_{\mathcal{F}}(I, QF) \tag{1.2}$$

The map from 1.1 is easily seen to be an isomorphism by the spectral sequence argument for the hyperext groups. We would like to analyse the possibility of 1.2 being an isomorphism via comparing both sides with $\oplus_i Ext^*_{\mathcal{F}}(I, L^s_i F)$. The present paper is the first step towards this. In Section 3 we obtain formulas relating $Ext^*_{\mathcal{F}}(F, I)$ and $\oplus_i Ext^*_{\mathcal{F}}(L^s_i F, I)$. The dual case is not so easy to handle and the reason is known. In this case the spectral sequence mentioned above with a chain complex as covariant variable does not have to converge. In [[10], 11.3] Stefan Schwede gives an example of a functor of infinite degree in $\mathcal{F}$, where R is a finite field and for which $\mathbf{Ext}^0(I, F)$ and $\mathbf{Ext}^0(I, QF)$ really differ. Hence our general theorem 4.1 is not fully satisfactory and cannot be stronger in full generality. We finish Section 4 with presenting some special situations when we get fully satisfactory answers also in this case.

The noetherianity assumption is not necessary for theorems 3.1 and 4.1 which should be true for any commutative ring R and functors from R-modules to R-modules, as stable derived functors are defined for functors between abelian categories. On the other hand at all places where we assume that our ring is a finite field we use machinery developed for functors from finite dimensional vector spaces to vector spaces. Because of these reasons we decided to restrict our attention to the case when finitely generated modules over R form an abelian category.

Conventions: We write **Ext** for the hyperext groups. Functors $F \in \mathcal{F}$ are treated as chain complexes with one nonzero object in dimension 0 which is equal to F. Hence functors can appear as variables in both $\mathbf{Ext}_{\mathcal{F}}$ and $Ext_{\mathcal{F}}$ groups. When we restrict our considerations to R being a finite field of characteristic p we write $\mathcal{F}_p$ instead of $\mathcal{F}$. Throughout the paper we assume that our functors are reduced ($F(0) = 0$).

2. Preliminaries

The Yoneda multiplication induces a ring structure on $Ext^*_{\mathcal{F}}(I, I)$ and, correspondingly, left and right module structures over this ring on $Ext^*_{\mathcal{F}}(I, F)$ and $Ext^*_{\mathcal{F}}(F, I)$ for any F. Because we are going to use the module structures of $Ext^*_{\mathcal{F}}(I, F)$ and $Ext^*_{\mathcal{F}}(F, I)$ over $Ext^*_{\mathcal{F}}(I, I)$ we need, as a tool, the following well known lemma from homological algebra (see [[9], Theorem 5.3]).

Lemma 2.1. *Let $0 \to A \to B \to C \to 0$ be an exact sequence in an abelian category $\mathcal{A}$ which contains enough projective and injective objects, D be a chosen object in $\mathcal{A}$ and $D^* = Ext^*_{\mathcal{A}}(D, D)$ be a graded ring with Yoneda multiplication. Then all maps in the long exact sequences*

$$... \to Ext^i_{\mathcal{A}}(C, D) \to Ext^i_{\mathcal{A}}(B, D) \to Ext^i_{\mathcal{A}}(A, D) \to Ext^{i+1}_{\mathcal{A}}(C, D) \to ...$$

and

$$... \to Ext^i_{\mathcal{A}}(D, A) \to Ext^i_{\mathcal{A}}(D, B) \to Ext^i(D, C)_{\mathcal{A}} \to Ext^{i+1}_{\mathcal{A}}(D, A) \to ...$$

are equivariant with respect to the action of D^* (right or left).

As a second tool for studying the Ext-groups of interest we are going to use the stable derived functors of functors between abelian categories. The left version of them was first defined by Dold and Puppe in [3], the right version was introduced in [11] which is a good reference for both notions. As an outcome, to any functor $F \in \mathcal{F}$ one associates two sequences $\{L^s_i F\}^{\infty}_{i=0}$ and $\{R^s_i F\}^{\infty}_{i=0}$ of left and right stable derived functors of F. They are additive and for us the crucial properties of these are contained in the following lemma (for the proof see [3, 11]).

Lemma 2.2. *(i). Let $0 \to F_1 \to F_2 \to F_3 \to 0$ be an exact sequence in $\mathcal{F}$. Then there are two long exact sequences of stable derived satellites*

$$... \to L^s_1 F_3 \to L^s_0 F_1 \to L^s_0 F_2 \to L^s_0 F_3 \to 0$$

$$0 \to R^s_0 F_1 \to R^s_0 F_2 \to R^s_0 F_3 \to R^s_1 F_1 \to ...$$

(ii). For $F \in \mathcal{F}$ there is a natural epimorphism $F \to L^s_0(F)$ and a natural monomorphism $R^s_0 F \to F$ which induce isomorphisms

$$Hom_{\mathcal{F}}(L^s_0 F, G) \to Hom_{\mathcal{F}}(F, G)$$

$$Hom_{\mathcal{F}}(G, R^s_0 F) \to Hom_{\mathcal{F}}(G, F)$$

for any additive $G \in \mathcal{F}$.

In the future, we will need some well known calculations of general nature. Recall that $F \in \mathcal{F}$ is called diagonalizable if it is a composition of the diagonal embedding $Mod_R \to (Mod_R)^d$ for some $d > 1$ with a functor $F' : (Mod_R)^d \to Mod_R$ which satisfies $F'(M_1, ..., M_d) = 0$ whenever there is an i such that $M_i = 0$. In the Appendix to [1] one can find the proof (by the second author) of:

Lemma 2.3. *If $F \in \mathcal{F}$ is diagonalizable then $Ext^*_{\mathcal{F}}(I, F) = Ext^*_{\mathcal{F}}(F, I) = 0$.*

One can also state the version of 2.3 for stable derived functors.

Lemma 2.4. *If $F \in \mathcal{F}$ is diagonalizable then $L^s_i F = R^s_i F = 0$ for $i \geq 0$.*

Proof. For $L^s_i F$ it follows immediately from [[3], Theorem 6.10]. The proof for $R^s_i F$ follows by the same method. $\square$

3. Generators for $Ext^*_{\mathcal{F}}(F,I)$

Let $... \xrightarrow{d_2} QF_2 \xrightarrow{d_1} QF_1 \xrightarrow{d_0} F \to 0$ be the cubical construction on F as studied in [[5], Sections 6 and 7]. It is a chain complex in $\mathcal{F}$ consisting of diagonalizable functors in degrees above 0 with homology functors isomorphic to the left stable derived functors of F. Lemma 2.3 implies that for any i the group $\mathbf{Ext}^i_{\mathcal{F}}(F,I)$ is isomorphic to $\mathbf{Ext}^i_{\mathcal{F}}(QF,I)$ by the obvious spectral sequence argument. Below we describe a filtration of $\mathbf{Ext}^*_{\mathcal{F}}(F,I)$ consisting of $\mathbf{Ext}^*_{\mathcal{F}}(I,I)$-submodules which gives a relation between $\mathbf{Ext}^*_{\mathcal{F}}(F,I)$ and the sum over i of $Ext^*_{\mathcal{F}}(L^s_iF,I)$. We obtain this filtration by truncating QF_*.

Theorem 3.1. *Let L^s_iF denote the ith stable derived functor of $F \in \mathcal{F}$. Then $Ext^*_{\mathcal{F}}(F,I)$ as a module over $Ext^*_{\mathcal{F}}(I,I)$ has an increasing filtration of graded submodules $0 = A^*_{-1} \subset A^*_0 \subset A^*_1 \subset A^*_2 \subset ...$ such that for any natural $n \geq 0$ the quotient A^*_n/A^*_{n-1} is a subquotient of $Ext^*_{\mathcal{F}}(L^s_nF,I)$ and*

$$Ext^*_{\mathcal{F}}(F,I) = \bigcup_{n=0}^{\infty} A^*_n.$$

Moreover, for any integer n, the quotient map gives an isomorphism in degree n between a subgroup of $Hom_{\mathcal{F}}(L^s_nF,I)$ and (A^n_n/A^n_{n-1}).

Proof. Let Q^n_* denote the homological truncation of the cubical complex QF_* at the level n. It means Q^n_* agrees with QF_* up to dimension n, $Q^n_{n+1} = im(d_{n+1})$ and $Q^n_i = 0$ for $i > n+1$. There is an obvious epimorphism $q^n : Q^n_* \to Q^{n-1}_*$ induced by identities up to dimension $n-1$ and d_n in dimension n. For every n there exists a morphism $i^n : F \to Q^n_*$ defined as identity in dimension 0. The morphisms q^n are compatible with i^n's ($q^n \circ i^n = i^{n-1}$) and for every n they induce a short exact sequences of chain complexes

$$0 \to K^n_* \longrightarrow Q^n_* \xrightarrow{q^n} Q^{n-1}_* \to 0 \tag{3.1.1}$$

The complex K^n_* is nontrivial only in dimensions n and $n+1$. The definition of q^n implies that $K^n_n = ker(d_n)$ and $K^n_{n+1} = im(d_{n+1})$. Hence by the obvious spectral sequence argument the hyperext groups with K^n_* coefficients satisfy

$$\mathbf{Ext}^i(K^n_*,I) = \mathbf{Ext}^i(L^s_nF[n],I)$$

Now we can define our filtration. Let $\mathbf{Ext}^*(i^n,I) : \mathbf{Ext}^*_{\mathcal{F}}(Q^n_*,I) \to \mathbf{Ext}^*_{\mathcal{F}}(F,I)$ be the map induced by i^n. We put

$$A^*_n = im(\mathbf{Ext}^*(i^n,I))$$

The formula $q^n \circ i^n = i^{n-1}$ implies that $A^*_{n-1} \subset A^*_n$. Observe that the kernel L^n_* of the projection $p^n : QF_* \to Q^n_*$ is trivial below dimension $n+1$ and hence $\mathbf{Ext}^i_{\mathcal{F}}(L^n_*,I) = 0$ for $i \leq n$. This implies by the long exact sequence of hyperext groups that $\mathbf{Ext}^k(i^n,I) : \mathbf{Ext}^k_{\mathcal{F}}(Q^n_*,I) \to \mathbf{Ext}^k_{\mathcal{F}}(F,I) \simeq \mathbf{Ext}^k_{\mathcal{F}}(QF,I)$ is an isomorphism for $k \leq n$. In conclusion we get that our filtration stabilizes

$$A^k_k = A^k_{k+1} = A^k_{k+2} = ... = \mathbf{Ext}^k_{\mathcal{F}}(F,I)$$

and then of course

$$Ext^*_{\mathcal{F}}(F,I) = \bigcup_{n=0}^{\infty} A^*_n.$$

Let now $C^n_k = coker(\mathbf{Ext}^k(q^n, I) : \mathbf{Ext}^k_{\mathcal{F}}(Q^{n-1}_*, I) \to \mathbf{Ext}^k_{\mathcal{F}}(Q^n_*, I))$. Immediately from the definition of our filtration we see that in a commuting diagram

$$\begin{array}{ccccccc}
\mathbf{Ext}^k_{\mathcal{F}}(Q^{n-1}_*, I) & \to & \mathbf{Ext}^k_{\mathcal{F}}(Q^n, I) & \to & C^k_n & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
A^k_{n-1} & \to & A^k_n & \to & A^k_n/A^k_{n-1} & \to & 0
\end{array}$$

all vertical maps are epimorphisms. On the other hand the long exact sequence of hyperext groups induced by the short exact sequence of complexes (3.1.1) implies that the group C^k_n is a subgroup of $\mathbf{Ext}^k_{\mathcal{F}}(L^s_n F[n], I)$. Observe that $\mathbf{Ext}^k_{\mathcal{F}}(L^s_n F[n], I) = Ext^{k-n}_{\mathcal{F}}(L^s_n F, I)$. Hence for $k = n$ we get that the cokernel C^n_n is a subgroup of $Hom_{\mathcal{F}}(L^s_n F, I)$. The final statement of the theorem comes from considering the spectral sequences for hyperext groups. Then it is obvious that A^n_n/A^n_{n-1} is equal to the kernel of all differentials starting from $Hom_{\mathcal{F}}(L^s_n F, I)$. □

Corollary 3.2. *Assume that for a given $F \in \mathcal{F}$, $L^s_i F = 0$ for any i. Then*

$$Ext^*_{\mathcal{F}}(F,I) = 0.$$

Proof. The assumption that all $L^s_i F = 0$ implies that all quotients in the filtration $A_*(F)$ are trivial. But then the filtration is trivial so $Ext^*_{\mathcal{F}}(F,I) = 0$. □

Remark 3.3. If R is a field then the conditions $L^s_i F = 0$ and $Ext^*_{\mathcal{F}}(F,I) = 0$ are equivalent. This comes from the fact that vanishing of $Ext^*_{\mathcal{F}}(F,I)$ implies triviality of the layers of the filtration $A_*(F)$. This observation easily implies that for any i, $Hom_{\mathcal{F}}(L^s_i F, I) = 0$. But this latter condition in our case implies $L^s_i F = 0$ because the *Hom*-set between nontrivial additive functors is never trivial.

For the rest of this section we assume that R is a finite field of characteristic p. To emphasize that we work under this assumption we will write $\mathcal{F}_p$ instead of $\mathcal{F}$. Observe that now the domain of our functors consists of finite dimensional vector spaces over R. Our goal is to get an analogous observation as in Theorem 3.1 for $Ext^*_{\mathcal{F}_p}(I,F)$ in this case. The answer will use the right stable derived functors of F, $R^s_* F$. Recall after [6] that in the case R is a finite field to any $F \in \mathcal{F}_p$ we can associate its Kuhn dual functor $F^{\#} \in \mathcal{F}_p$ defined for any finite dimensional vector space V by the formula

$$F^{\#}(V) = (F(V^{\bullet}))^{\bullet},$$

where upper $\bullet$ denotes the ordinary linear dual. We will call a functor *finite* if it is of finite degree and has values in finite dimensional vector spaces. We have the following lemma (compare [[6], Proposition 4.4]):

Lemma 3.4. *For any i and finite F and G we have:*

$$Ext^i_{\mathcal{F}_p}(F,G) = Ext^i_{\mathcal{F}_p}(G^{\#}, F^{\#}).$$

The functor I is self dual and the lemma above gives us a useful identification of groups $Ext^i_{\mathcal{F}_p}(I, F)$ and $Ext^i_{\mathcal{F}_p}(F^\#, I)$ for any finite F. This identification agrees with the Yoneda multiplication and hence $Ext^*_{\mathcal{F}_p}(I, F)$ and $Ext^*_{\mathcal{F}_p}(F^\#, I)$ are isomorphic as $Ext^*_{\mathcal{F}_p}(I, I)$-modules. This observation and Theorem 3.1 give the following:

Corollary 3.5. *For any finite $F \in \mathcal{F}$ the graded group $Ext^*_{\mathcal{F}_p}(I, F)$ as a module over $Ext^*_{\mathcal{F}}(I, I)$ has an increasing filtration $0 = A^*_{-1} \subset A^*_0 \subset A^*_1 \subset A^*_2 \subset \dots$ such that for $n \geq 0$ the quotient A^*_n/A^*_{n-1} is a subquotient of $Ext^*_{\mathcal{F}_p}(L^s_n(F^\#), I)$ and*

$$Ext^*_{\mathcal{F}_p}(I, F) = \bigcup_{n=0}^{\infty} A^*_n.$$

Moreover the quotient map gives an isomorphism in degree n from (A^n_n/A^n_{n-1}) to a subgroup of $Hom_{\mathcal{F}_p}(L^s_n(F^\#), I)$.

We can translate statement of the corollary above using the following lemma:

Lemma 3.6. *For any finite $F \in \mathcal{F}_p$ and any natural i we have a natural isomorphism*

$$L^s_i(F^\#) \simeq (R^s_i F)^\#$$

Proof. The isomorphism comes directly from the definition of stable derived functors. Shortly: let A be a finite dimensional vector space. We know that for any $F \in \mathcal{F}_p$ and $i < 2n$ we have

$$L^s_i F(A) = \pi_{n+i}(F(A_*(n))),$$

where $A_*(n)$ is a simplicial (projective) resolution of A of degree n. Typically we construct $A_*(n)$ using spheres. Let S^n denote the reduced simplicial model of the n-dimensional sphere with no simplices below n. We can take $A_*(n)$ to be the A-free simplicial vector space $A[S^n]$. In order to get $R^s_i F(A)$ we should take a cosimplicial (injective) resolution of A of degree n, apply to it functor F and calculate cohomology groups of the corresponding cochain complex. But observe that for any finite set X, $(A[X])^\bullet$ is naturally isomorphic to $A^\bullet[X]$ and hence for the set S^n_i of i dimensional simplices of a sphere

$$(A[S^n_i])^\bullet = A^\bullet[S^n_i]$$

This implies that $(A[S^n])^\bullet$ is a cosimplicial resolution of $A^\bullet$ of degree n. We can now calculate:

$$L^s_i(F^\#)(A) = \pi_{n+i} F^\#(A[S^n]) = \pi_{n+i}(F(A[S^n]^\bullet)^\bullet =$$
$$\pi_{n+i}(F(A^\bullet[S^n])^\bullet = (H^{n+i}(F(A^\bullet[S^n]))^\bullet = (R^s_i F)^\#(A)$$

□

Remark 3.7. When F has values in finite dimensional vector spaces then $R^s_i F$ is self dual and we get the filtration of $Ext^*_{\mathcal{F}_p}(I, F)$ governed by the groups $Ext^*_{\mathcal{F}_p}(I, R^s_i F)$. We suspect that this statement is true in greater generality but for the proof we need a nonnegative cochain complex $\bar{Q}F^*$ in $\mathcal{F}$ with $\bar{Q}F^0 = F$, diagonalizable $\bar{Q}F^i$ for $i > 0$ and cohomology functors isomorphic to $R^s_* F$. Of course, in the case of finite fields and finite functors, it is easy to check that the cochain complex $(QF_*^{\#})^{\#}$ has the desired properties.

4. Generators for $Ext^*(I, F)$ via $L_*^s F$

We start the present section from proving some general version of theorem 3.1 for $Ext^*_{\mathcal{F}}(I, F)$. The new problem arises when one wants to show that the constructed filtration is exhausting.

Theorem 4.1. *Let $F \in \mathcal{F}$. The graded group $Ext^*_{\mathcal{F}}(I, F)$ as a module over $Ext^*_{\mathcal{F}}(I, I)$ has a decreasing filtration $Ext^*_{\mathcal{F}}(I, F) = B^*_{-1} \supset B^*_0 \supset B^*_1 \supset B^*_2 \supset ...$ such that for $-1 \leq n$ the quotient B^*_n / B^*_{n+1} is a subquotient of $Ext^*_{\mathcal{F}}(I, L^s_{n+1} F)$.*

Proof. We will use notation from the proof of 3.1. We put $B^*_{-1} = Ext^*_{\mathcal{F}}(I, F)$ and $B^*_n = ker(\mathbf{Ext}^*(I, i^n) : \mathbf{Ext}^*_{\mathcal{F}}(I, F) \longrightarrow \mathbf{Ext}^*_{\mathcal{F}}(I, Q^n_*))$. From the equality $q^n \circ i^n = i^{n-1}$ one gets that $B^*_n \subset B^*_{n-1}$. Let D^k_n denote the cokernel of the embedding $B^k_n \hookrightarrow \mathbf{Ext}^k_{\mathcal{F}}(I, F)$. Then for any k we have a commuting diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & B^k_n & \to & \mathbf{Ext}^k_{\mathcal{F}}(I, F) & \to & D^k_n & \to & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \to & B^k_{n-1} & \to & \mathbf{Ext}^k_{\mathcal{F}}(I, F) & \to & D^k_{n-1} & \to & 0 \end{array}$$

By the Snake Lemma we get an isomorphism between B^k_{n-1}/B^k_n and the kernel X^k_n of $D^k_n \to D^k_{n-1}$. It follows immediately from the definition of D^*_n that X^k_n embeds into the kernel of $\mathbf{Ext}^k(I, q^n)$. From the the long exact sequence of hyperext groups induced by the exact sequence 3.1.1 we read that $\mathbf{Ext}^k_{\mathcal{F}}(I, L^s_n F[n])$ maps onto the kernel of $\mathbf{Ext}^k(I, q^n)$. This finishes the proof of the theorem. □

For computational purposes we would like to have an exhausting filtration. It means here that we would like to have the formula:

$$\bigcap_{n \geq 0} B^*_n = 0. \tag{4.1.1}$$

Unfortunately we cannot claim this. We know that this formula is not true in general. Schwede's example ([[10], Example 11.3]), mentioned in the introduction, gives a counterexample to it. His functor F has the property that all of its left stable derived functors are trivial but $Hom_{\mathcal{F}}(I, F)$ is nontrivial. We suspect that 4.1.1 holds for finite degree functors. Below we discuss the special case, when we can prove that the filtration described above is exhausting. For this we have to assume (as at the end of Section 3) that our ring R is a finite field of characteristic p. We send the interested reader to [4] for the definition and properties of the category $\mathcal{P}$ of strict polynomial functors. It comes with the forgetful functor $\iota : \mathcal{P} \to \mathcal{F}_p$. We will use in the future the same letter for denoting $F \in \mathcal{P}$ and its image $\iota(F) \in \mathcal{F}_p$. Our goal now is to prove the following theorem:

Theorem 4.2. *Let $F \in \mathcal{F}_p$ be in the image of ι. Then the filtration described in Theorem 4.1 on $Ext^*_{\mathcal{F}_p}(I, F)$ is exhausting.*

Before the proof, be would like to make some general comments and observations. Every functor $F \in \mathcal{P}$ decomposes into a sum of its homogeneous pieces. The forgetful functor ι carries sums to sums. Moreover Ext groups are additive with respect to both variables and the decomposition $Ext^*_{\mathcal{F}_p}(I, F_1 \oplus F_2) = Ext^*_{\mathcal{F}_p}(I, F_1) \oplus Ext^*_{\mathcal{F}_p}(I, F_2)$ is valid in the category of $Ext^*_{\mathcal{F}_p}(I, I)$-modules. The Q-construction is additive with respect to the functor and so are the stable left derived functors. All this means that we can assume that our functor F comes from a homogeneous functor of degree t. The crucial observation for proving 4.2 is contained in the proposition below. It should potentially be interesting and fruitful on its own because it shows that the duality phenomenon relating $Ext^*_{\mathcal{F}_p}(I, S^t)$ and $Ext^*_{\mathcal{F}_p}(I, D^t)$ (symmetric and divided power functors), which we trivially prove using Koszul and de Rham sequences, applies to all finite degree homogeneous functors F and their Kuhn duals $F^{\#}$.

Proposition 4.3. *Let $F \in im(\iota)$ be a functor of homogeneous degree t and assume that $|R| \geq t$. Then for any given i there exists s such that $Ext^i_{\mathcal{F}_p}(I, F)$ and $Ext^{i+s}_{\mathcal{F}_p}(F, I)^{\bullet}$ are naturally isomorphic.*

Proof. The functor F comes from the category $\mathcal{P}$ so it has in $\mathcal{P}$ a finite injective resolution $F \to Q_0 \to Q_1 \to ...$ in which every Q_i is a sum of tensor products of symmetric powers:

$$Q_i = \bigoplus S^{j_1} \otimes ... \otimes S^{j_k}$$

where $j_1 + ... + j_k = t$. This resolution remains exact when considered in $\mathcal{F}$. If $k > 1$ then $S^{j_1} \otimes ... \otimes S^{j_k}$ is obviously diagonalizable. If $k = 1$ and j_1 is not a power of p then S^{j_1} is a direct summand in a diagonalizable functor (see [4], Proposition 6.1]). Hence by a hyper-cohomology spectral sequence argument and Lemma 2.3 we know that $Ext^*_{\mathcal{F}_p}(I, F)$ is non trivial only when t is a power of p. We will assume that $t = p^d$ for the rest of the proof.

If $|R| > t$ then there is nothing to prove because both sides of the equation are 0 by [[7], Section 5.2]. So for the rest of the proof we assume that $|R| = t$. When this is the case then $I = I^{(d)}$ and we will use this equality in the future. By [[2], Theorem 3.2] we know that $Ext^i_{\mathcal{P}}(I^{(d)}, F) = Ext^{i+s}_{\mathcal{P}}(I^{(d)}, F^{\#})^{\bullet}$ where s depends only on the degree of F. Here $(.)^{(d)}$ denotes traditionally the precomposition of a functor with the d-fold composition of the Frobenius twist. For any $F, G \in \mathcal{P}$ the canonical homomorphism $Ext^i_{\mathcal{P}}(F, G) \to Ext^i_{\mathcal{F}_p}(F, G)$ induced by ι is a monomorphism ([[4], Corollary 3.7]). Assume now that $R \to K$ is a finite field extension. We will write $\mathcal{F}(K)$ ($\mathcal{P}(K)$) for the corresponding categories of functors over K. Let $F_K \in \mathcal{P}(K)$ be the functor obtained from F by the change of the base field (see [[4], Proposition 1.1]). We have formulas

$$K \otimes Ext^*_{\mathcal{F}_p}(I, F) = Ext^*_{\mathcal{F}(K)}(I^{(d)}, F_K)$$

$$K \otimes Ext^*_{\mathcal{P}}(I^{(d)}, F) = Ext^*_{\mathcal{P}(K)}(I^{(d)}, F_K)$$

The second equation is stated in [[4], Proposition 1.1] while the first one follows from [[4], Theorem 3.9] and equality $I = I^{(d)}$, which holds over R. Of course the same formulas hold for the groups $Ext(.,I)$. Theorem 3.10 of [4] plus Corollary 2.8 in the same paper tell us that when K is large enough and i is given then there exists m_0 such that for all $m \geq m_0$ we have an isomorphism:

$$Ext^i_{\mathcal{P}(K)}(I^{(m+d)}, F_K^{(m)}) \simeq Ext^i_{\mathcal{F}(K)}(I, F_K)$$

We can assume that for $m \geq m_0$ holds also:

$$Ext^i_{\mathcal{P}(K)}(F_K^{(m)}, I^{(m+d)}) \simeq Ext^i_{\mathcal{F}(K)}(F_K, I)$$

From these we deduce that for a certain sufficiently large K we have:

$$K \otimes Ext^i_{\mathcal{F}_p}(I,F) \simeq Ext^i_{\mathcal{F}(K)}(I^{(d)}, F_K) \simeq Ext^i_{\mathcal{P}(K)}(I^{(m+d)}, F_K^{(m)}) \simeq$$

$$\simeq K \otimes Ext^i_{\mathcal{P}}(I^{(m+d)}, F^{(m)}) \simeq K \otimes Ext^{i+s}_{\mathcal{P}}(I^{(m+d)}, F^{(m)\#})^{\bullet} \longleftarrow$$

$$\longleftarrow K \otimes Ext^{i+s}_{\mathcal{F}_p}(I, F^{\#})^{\bullet}.$$

where the last arrow is an epimorphism. The first five isomorphisms follow from our discussion proceeding the statement and the number s is determined by the degree of $F^{(m)}$. But in order to get the isomorphisms on the last step we might have to twist functors again. The last arrow is dual to the arrow

$$Ext^{i+s}_{\mathcal{P}}(I^{(m+d)}, F^{(m)\#}) \to Ext^{i+s}_{\mathcal{P}}(I^{(m+d+e)}, F^{(m+e)\#}) \to K \otimes Ext^{i+s}_{\mathcal{F}_p}(I, F^{\#}).$$

for a certain number e where the first map is a monomorphism and the second is an isomorphism. From this we get that $Ext^{i+s}_{\mathcal{F}_p}(I, F^{\#})^{\bullet}$ maps onto $Ext^i_{\mathcal{F}_p}(I,F)$. But now we can argue the same way but starting from $F^{\#}$ instead of F. This way we show that our epimorphism is also a monomorphism. Finally, the naturality statement follows from [[4], Proposition 1.4]. □

Corollary 4.4. *Let $F \in im(\iota)$ be a functor of homogeneous degree t. Then for any given i there exists s such that $Ext^i_{\mathcal{F}_p}(I,F)$ and $Ext^{i+s}_{\mathcal{F}_p}(F,I)^{\bullet}$ are isomorphic.*

Proof. We have to show our statement only for $t = p^d$ and $|R| < p^d$. We will write the argument only for the based field $R = F_p$ leaving the case of the general R like above for the reader. Let K be a degree d extension of F_p. Let τ be a functor of restriction of scalars from K to F_p and ρ denotes the scalar extension. By [[4], Remark 3.4.1] we know that

$$K \otimes Ext^*_{\mathcal{F}_p}(I,F) \simeq Ext^*_{\mathcal{F}(K)}(I \circ (\rho \circ \tau), F_K)$$

By [[7], Section 5.2] we know that the latter group is isomorphic to

$$Ext^*_{\mathcal{F}(K)}(I, F_K).$$

Hence we finish our proof using the diagram

$$\begin{array}{ccc} K \otimes Ext^i_{\mathcal{F}_p}(I,F) & \simeq & Ext^i_{\mathcal{F}(K)}(I, F_K) \\ & & \downarrow \\ K \otimes Ext^{i+s}_{\mathcal{F}_p}(F,I)^{\bullet} & \simeq & Ext^{i+s}_{\mathcal{F}(K)}(F_K, I)^{\bullet} \end{array}$$

where the right vertical arrow is an isomorphism from 4.3. □

Before proving 4.2 we need one more observation. If F is a homogeneous functor in $\mathcal{P}$ of degree $t = p^d$ then for every n the functor QF_n is strict polynomial of degree t and the complex QF_* is a complex of objects in $\mathcal{P}$. From this point of view the stable derived functors (homology of QF_*) are additive strict polynomial functors of degree t. The theorems 3.1, 4.1 and 4.2 are valid in $\mathcal{P}$ and their proofs are the same, of course with $I^{(d)}$ at the place of I. Moreover in $\mathcal{P}$ we do not have problems with proving equality between $Ext^*_{\mathcal{P}}(I^{(d)}, F)$ and $\mathbf{Ext}^*_{\mathcal{P}}(I^{(d)}, F)$. Let $\mathcal{P}_t$ denote the category of strict polynomial functors of degree t. Then $\mathcal{P}_t$ has finite cohomological dimension and hence the hyperext spectral sequence strongly converges. On the other hand we know that the groups $Ext^*_{\mathcal{P}}(I^{(d)}, F)$ are the same as $Ext^*_{\mathcal{P}_t}(I^{(d)}, F)$.

Proof of Theorem 4.2. Let F be as in the theorem. We can assume that F is homogeneous of degree $t = p^d$. Moreover assume that $|R| = t$. This implies that we have the full naturality of the isomorphism from Proposition 4.3. Our functor takes finite dimensional vector spaces to finite dimensional vector spaces and hence the linear duality operation is contravariant exact. Moreover for such functors all Ext-groups between them are finite dimensional vector spaces.

We have to show that if $\alpha \in Ext^i_{\mathcal{F}}(I, F)$ then there exists j such that $\alpha \in B^i_j$ and $\alpha \notin B^i_{j+1}$. By 4.3 we know that there exists s such that we have the following isomorphism of Ext-groups:

$$Ext^i_{\mathcal{F}_p}(I, F) \simeq Ext^{i+s}_{\mathcal{F}_p}(I, F^{\#})^{\bullet} \simeq Ext^{i+s}_{\mathcal{F}_p}(F, I)^{\bullet}$$

Assume that the isomorphisms above maps α to $\bar{\alpha}^{\bullet}$ for some $\bar{\alpha} \in Ext^{i+s}_{\mathcal{F}_p}(F, I)$. To illustrate our approach assume first that $\bar{\alpha} \in A^*_0$, where A^*_0 is as in the theorem 3.1. Recall that by the definition of our truncations $\mathbf{Ext}^*_{\mathcal{F}_\mathbf{p}}(\mathbf{Q^0_*}, \mathbf{I}) = \mathbf{Ext}^*_{\mathcal{F}_\mathbf{p}}(\mathbf{L^s_0 F}, \mathbf{I})$. We can extend the sequence of isomorphisms described above to the commuting diagram

$$\begin{array}{ccccc} Ext^i(I,F) & \simeq & Ext^{i+s}(I,F^{\#})^{\bullet} & \simeq & Ext^{i+s}(F,I)^{\bullet} \\ \downarrow & & \downarrow & & \downarrow \\ Ext^i(I,L^s_0 F) & \simeq & Ext^{i+s}(I,(L^s_0 F)^{\#})^{\bullet} & \simeq & Ext^{i+s}(L^s_0 F,I)^{\bullet} \end{array} \tag{4.1.2}$$

in which the image of α in right lower corner is nontrivial. This means that α is not in the kernel of the left vertical map and hence is not in B^*_0.

In the general case we proceed similarly using notation from the proof of 3.1. Let $0 \neq \bar{\alpha} \in A^*_n \setminus A^*_{n-1}$. From the exact sequence

$$0 \to F \to Q^{n-1} \to Q^{n-1}/F \to 0$$

we know that $\bar{\alpha}$ determines a nontrivial element in

$$\mathbf{Ext}^{i+s+1}_{\mathcal{F}_p}(Q^{n-1}/F, I) = \mathbf{Ext}^{i+s+1}_{\mathcal{F}_p}(K^{n-1}_n[n], I) = \mathbf{Ext}^{i+s+1-n}_{\mathcal{F}_p}(K^{n-1}_n, I)$$

by the boundary homomorphism. Finally, by the short exact sequence of functors

$$0 \to K^n_n \to Q^n_n \to K^{n-1}_n \to 0$$

and diagonalizability of Q^n_n we get a nontrivial element

$$\tilde{\alpha} \in \mathbf{Ext}^{i+s}_{\mathcal{F}_p}(K^n_n[n], I) = \mathbf{Ext}^{i+s-n}_{\mathcal{F}_p}(K^n_n, I).$$

On the other hand $\bar{\alpha}$ is in the image of $\mathbf{Ext}(i^n, I)$ so in the long exact sequence of $\mathbf{Ext}$-groups related to the short exact sequence

$$0 \to K^n_n[n] \to K^n_* \to K^n_{n+1}[n+1] \to 0$$

$\tilde{\alpha}$ determines a nontrivial element in $\mathbf{Ext}^{i+s}_{\mathcal{F}_p}(K^n_*, I) = \mathbf{Ext}^{i+s}_{\mathcal{F}_p}(L^s_n F[n], I)$. By our knowledge about stable derived functors we see that $L^s_0(K^n_n) = L^s_n F$ and the map $K^n_n[n] \to K^n_*$ induces the same map on the $\mathbf{Ext}$-groups (with corresponding shift) as $K^n_n \to L^s_0(K^n_n)$.

Assume that $\alpha \in B^*_{n-1}$ and $\bar{\alpha} \in A^*_n \setminus A^*_{n-1}$. By the same reasoning as above (but covariant) we see that α determines an element $\underline{\alpha} \in \mathbf{Ext}^i_{\mathcal{F}_p}(I, K^n_n[n])$. By the naturality of 4.3 we know that we can make choices in such a way that $\underline{\alpha}$ is mapped to $(\tilde{\alpha})^\bullet$ in the isomorphism from 4.3. Now we can use the commuting diagram

$$\begin{array}{ccc} Ext^i(I, K^n_n) & \simeq & Ext^{i+s}(K^n_n, I)^\bullet \\ \downarrow & & \downarrow \\ Ext^i(I, L^s_n F) & \simeq & Ext^{i+s}((L^s_n F), I)^\bullet \end{array}$$

to show that $\alpha \notin B^*_n$. Note that the diagram above is the same as the diagram 4.4.1 with K^n_n instead of F. Of course we identify $L^s_0(K^n_n) = L^s_n F$.

For the general R we use Corollary 4.4. As in 4.4 we write our argument only for F_p. Assume that we extend our field of scalars $F_p \to K$ with $|K| = p^d$. Obviously $Q(F_K)_* = (QF_*)_K$ and for any natural n the map i^n and q^n commute with the extension of scalars. It means that extension of scalars commutes with filtrations. We get commuting diagrams

$$\begin{array}{ccc} K \otimes A^*_n(F) & \hookrightarrow & K \otimes Ext^*_{\mathcal{F}_p}(F, I) \\ \downarrow & & \downarrow \\ A^*_n(F_K) & \hookrightarrow & Ext^*_{\mathcal{F}_K}(F_K, I) \end{array}$$

and

$$\begin{array}{ccc} K \otimes B^*_n(F) & \hookrightarrow & K \otimes Ext^*_{\mathcal{F}_p}(I, F) \\ \downarrow & & \downarrow \\ B^*_n(F_K) & \hookrightarrow & Ext^*_{\mathcal{F}_K}(I, F_K) \end{array}$$

The right vertical arrows are isomorphisms hence the left vertical maps are monomorphisms. The filtration $B^*_n(F_K)$ is exhausting. This implies that the filtration $B^*_n(F)$ is also exhausting. □

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