HOMOLOGICAL ALGEBRA IN THE CATEGORY OF Γ-MODULES

by

Jakub Antosz and Stanislaw Betley*

ABSTRACT: We study homological algebra in the abelian category $\tilde{\Gamma}$, whose objects are functors from finite pointed sets to vector spaces over F_p. The full calculation of $Tor_*^{\tilde{\Gamma}}$-groups between functors of degree not exceeding p is presented. We compare our calculations with known results on homology of symmetric groups, Steenrod algebra and functor homology computations in the abelian category $\mathcal{F}$ of functors from vector spaces over F_p to itself.

0. Introduction.

In recent years we observe growing interest in homological algebra computations in various categories of functors from small categories to vector spaces. Let Γ be the category of finite pointed sets. By Γ-module we understand a functor from Γ to vector spaces over a finite field F_p. The following paper is the first in a series devoted to studying homological algebra in the category of Γ-modules, which will be denoted by $\tilde{\Gamma}$. The homological algebra in the category $\tilde{\Gamma}$ is of crucial importance because of its close relations to Steenrod algebra and algebraic topology in general. The subject is well documented in the literature, see for example [BS], [B1], [B2], [P1], [Ri], [Ro] etc.

If we denote by $\mathcal{V}_p$ the category of finite dimensional vector spaces over F_p and by $\mathcal{F}$ the category of functors from $\mathcal{V}_p$ to $Vect_{F_p}$ then one can say that homological algebra in $\mathcal{F}$ is well understood because of calculations and methods developed during the last ten years with a culmination in [FFSS]. But some questions still remain open. Let $L \in \tilde{\Gamma}$ be a linearization functor which takes a pointed set X with a distinguished point 0 to $F_p[X]/F[0]$. Categories $\tilde{\Gamma}$ and $\mathcal{F}$ are related by the functor $l : \mathcal{F} \to \tilde{\Gamma}$ via the formula $l(T) = T \circ L$ and hence their homological algebras are also related. This correspondence was preliminary studied in [B2] where it was shown how to apply $\tilde{\Gamma}$-calculations to obtain new interesting results in $\mathcal{F}$. It seems to us that homological algebra in $\tilde{\Gamma}$ should be easier than in $\mathcal{F}$ and the full knowledge on both should come from their interaction coming from the functor l.

We will use the following convention: we will denote by the same letter a functor from $\mathcal{F}$ and its precomposition with L. This should not cause any problem in the present paper because the category $\mathcal{F}$ will not be used here in any systematic way. If we want to get from $T \in \mathcal{F}$ a contravariant functor $\Gamma \to Vect_{F_p}$ we will precompose it with L^* where $*$ denotes the ordinary vector space dualization. In such notation we can say that our ultimate goal is to get full understanding of the $Tor^{\tilde{\Gamma}}$ and $Ext_{\tilde{\Gamma}}$ groups between functors of exterior (Λ^i), symmetric (S^i) and divided powers (D^i), parallelly to the results of [FFSS]. Notice that

* Partially supported by the KBN grant 1P03A 005 26

[FFSS] is lacking calculations of $Ext_{\mathcal{F}}$ groups from the functors of symmetric powers to exterior or divided ones. The authors of [FFSS] do not see the systematic approach to such calculations. In the present note we give in 5.6 the full computation of $Tor_*^{\tilde{\Gamma}}(S^p, \Lambda^p)$ (why Tor instead of Ext is explained at the end of Introduction).

The prime number p is fixed in the whole paper and we will assume that it is not equal to 2. Our calculations can be done also for $p = 2$ but then some formulas are different. On the other hand this case can be treated by the same methods, so we leave it for the interested reader. The paper is organized as follows. In section 1 we will review known definitions, results and methods for the homological calculations in $\tilde{\Gamma}$. One can find there also some useful spectral sequences ant their applications. Some important calculations of Tor-groups are also there. Section 2 contains a discussion on similarities and differences between $\mathcal{F}$ and $\tilde{\Gamma}$ situation. Sections 3, 4 and 5 contain calculations of the homological algebra in $\tilde{\Gamma}$ between functors of degrees not exceeding p. As a side effect we get here a simple calculation of $H_*(\Sigma_p, F_p)$, which we are going to extend to all Σ_n in the next paper.

For shortness, the paper contains only results about Tor-groups. The calculations are very formal so translating them to the Ext-situation should not cause any problems. One remark is here in order: in $Tor(F, G)$ the first variable has to be contravariant and the second covariant. Hence in the case of Tor-groups we have only one calculation. In the Ext-case we have two situations which differ by variancy and which give different results. But the ingredients for the calculations remain the same.

I. Preliminaries.

Let us start from recalling the basic notation which will be used throughout the paper. Γ denotes the category of finite pointed sets. The typical object is given by the set $[n] = \{0, 1, ..., n\}$ where 0 is a base point. Of our primary interest is the category $\tilde{\Gamma}$ of functors from Γ to $Vect_{F_p}$ - vector spaces over the prime field F_p. All functors on vector spaces over F_p will be viewed as objects in $\tilde{\Gamma}$ via the precomposition with the linearization functor L or its dual L^*. The category $\tilde{\Gamma}$ contains enough projective objects. Among them are projective generators of $\tilde{\Gamma}$

$$\Gamma^n = F_p[Hom_\Gamma([n], -)]$$

Similarly,

$$\Gamma_n = F_p[Hom_\Gamma(-, [n])]$$

are projective generators in $\tilde{\Gamma}^{op}$, the category of contravariant functors from Γ to $Vect_{F_p}$. By Yoneda's lemma we have natural equivalences

$$F \otimes_\Gamma \Gamma^n = F(n)$$

and

$$\Gamma_n \otimes_\Gamma G = G(n)$$

While studying homological algebra in $\tilde{\Gamma}$ we will use very often the beautiful theorem of Pirashvili from [P2], which allows to do calculations in the much smaller category. Let Ω denote the category of finite sets and surjections. The typical object here will be denoted by $< n >= \{1, ..., n\}$. Let $\tilde{\Omega}$ denote the category of Ω-modules over F_p. The Dold-Kan theorem in Pirashvili's version tells us that categories $\tilde{\Gamma}$ and $\tilde{\Omega}$ are equivalent and homological algebras in them are the same. The equivalence is given by the so-called cross-effect functor $cr : \tilde{\Gamma} \to \tilde{\Omega}$. Observe, that all functors in which we are interested are of finite degree which means that cr takes them to a finite sequence of modules over F_p-group rings of certain symmetric groups. Every such object in $\tilde{\Omega}$ has a finite filtration (ascending in $\tilde{\Omega}$ and descending in $\tilde{\Omega}^{op}$) with quotients concentrated only on one object of Ω. We shall call such objects *atomic*. Observe that an atomic object concentrated on k is given by a Σ_k-module. By general homological algebra methods using spectral sequences we can reduce calculations of

$$Tor_*^{\tilde{\Omega}}(F, G)$$

to calculations of

$$Tor_*^{\tilde{\Omega}}(M, N)$$

where M and N are atomic.

The Tor-groups between atomic functors in $\tilde{\Omega}$ were calculated in [BS, formula 2.12] and the answer is given in terms of homology of subgroups of products of symmetric groups. So generally speaking all

$$Tor_*^{\tilde{\Omega}}(F, G)$$

for finite degree F and G are calculable. But of course in practice this is not the case, because usually we only get an answer encoded in a series of spectral sequences. One of the reasons for writing this note was to develop ways of calculating $Tor_*^{\tilde{\Omega}}(F, G)$ without using any knowledge from the (co)homology of symmetric groups.

Convention: We will denote $F \in \tilde{\Gamma}$ and its image in $\tilde{\Omega}$ (via cr) with the same letter. Hence it make sense for example to say that an object of $\tilde{\Gamma}$ is atomic. For example: Λ^i is atomic for any i because $cr(\Lambda^i)$ is concentrated on i and is equal to the sign permutation of Σ_i.

Observe that all functors which we are really interested in come from the tensor products of L or L^* with itself by dividing (or taking fixed points) by some action of symmetric group. Hence first step is to understand well the Tor-groups where one has certain tensor product of the linearization functor as one of the variables. The functor L is a direct summand of Γ^1 so it is projective and its tensor powers are projective as well. Hence we really have to start from calculating Tor-groups with tensor product of L^* as the contravariant variable. This will be accomplished by the end of this section. The methods of achieving this will come from studying the inner tensor products in $\tilde{\Gamma}$. The discussion below (up to 1.2) is taken, essentially, from [P1, section 4.1], so we are rather sketchy here.

The category $\tilde{\Gamma}$ is equipped with two types of inner tensor product. Let A, B and C, D be respectively objects of $\tilde{\Gamma}^{op}$ and $\tilde{\Gamma}$. Then we have two versions of inner tensor product $\odot$, each of them making following functorial isomorphisms true:

$$(A \otimes B) \otimes_\Gamma D \cong A \otimes_\Gamma (B \odot_I D)$$

$$A \otimes_\Gamma (C \otimes D) \cong (A \odot_{II} C) \otimes_\Gamma D$$

So we have defined two functors:

$$\odot_I : \tilde{\Gamma}^{op} \times \tilde{\Gamma} \to \tilde{\Gamma}$$

$$\odot_{II} : \tilde{\Gamma}^{op} \times \tilde{\Gamma} \to \tilde{\Gamma}^{op}$$

Thanks to Yoneda lemma we can recover explicit formulas for these products putting above $A = \Gamma_n$ and $D = \Gamma^n$, and getting respectively:

$$(B \odot_I D)[n] \cong (\Gamma_n \otimes B) \otimes_\Gamma D$$

$$(A \odot_{II} C)[n] \cong A \otimes_\Gamma (C \otimes \Gamma^n)$$

We can perform this kind of construction in any category of functors, but in Γ-modules it might be rewritten in slightly different form, because category Γ admits sums and products. For any U from the category $\tilde{\Gamma}$ or $\tilde{\Gamma}^{op}$ we have functors Δ_n and $\widetilde{\Delta}_n$ which don't change variancy.

$$(\Delta_n U)[m] := U([n] \times [m])$$

$$(\widetilde{\Delta}_n U)[m] := U([n] \vee [m])$$

There are natural isomorphisms:

$$(\Gamma_n \otimes B) \otimes_\Gamma D \cong B \otimes_\Gamma \Delta_n D$$

$$A \otimes_\Gamma (C \otimes \Gamma^n) \cong \widetilde{\Delta}_n A \otimes_\Gamma C$$

It is easy to see that these formulas are true for $B = \Gamma_m$ and $C = \Gamma^m$ thanks to:

$$[n] \vee [m] \cong [n+m]$$

$$[n] \times [m] \cong [nm + n + m]$$

leading to the isomorphisms in $\tilde{\Gamma}$:

$$\Gamma^n \otimes \Gamma^m \cong \Gamma^{n+m}$$

$$\Gamma_n \otimes \Gamma_m \cong \Gamma_{nm+n+m}$$

Following lemma tells us that it is enough.

Lemma 1.1: (Yoneda principle) Let $G_1, G_2 : \tilde{\Gamma} \to Vect_{F_p}$ be right exact functors commuting with sums. Let $\widetilde{G}_i$ be the composition $\Gamma^{op} \to \tilde{\Gamma} \to vect_{F_p}$ where first arrow assigns Γ^n to $[n]$, latter being just G_i. If $\widetilde{G}_1 \cong \widetilde{G}_2$ then $G_1 \cong G_2$.

It is obvious that we have well defined objects : $\mathbf{Tor}^I_*(B, D) \in \tilde{\Gamma}$ and $\mathbf{Tor}^{II}_*(A, C) \in \tilde{\Gamma}^{op}$ which are left derived functors of $\odot_I$ and $\odot_{II}$ respectively:

$$\mathbf{Tor}^I_*(B, D)[n] \cong Tor^{\tilde{\Gamma}}_*(B, \Delta_n D)$$

$$\mathbf{Tor}^{II}_*(A, C)[n] \cong Tor^{\tilde{\Gamma}}_*(\widetilde{\Delta}_n A, C)$$

Again by the general homological algebra methods we can show

Lemma 1.2: There exist two spectral sequences of composition of functors:

$${}^I E^2_{i,j} = Tor^{\tilde{\Gamma}}_i(A, \mathbf{Tor}^I_j(B, D)) \Rightarrow Tor^{\tilde{\Gamma}}_{i+j}(A \otimes B, C)$$

$${}^{II} E^2_{i,j} = Tor^{\tilde{\Gamma}}_i(\mathbf{Tor}^{II}_j(A, C), D) \Rightarrow Tor^{\tilde{\Gamma}}_{i+j}(A, C \otimes D)$$

Let us finish this section with sample applications of 1.2, related to our main object of study. This will lead us to the full understanding of $Tor^{\tilde{\Gamma}}_i(L^{*\otimes a}, \Lambda^p)$.

We will use following fact which might be found in [P1, Theorem 2.2 and Lemma 4.2].

Lemma 1.3: For any Γ-modules F, T and $i \geq 0$ we have an isomorphism:

$$Tor^{\tilde{\Gamma}}_i(L^*, F \otimes T) \cong Tor^{\tilde{\Gamma}}_i(L^*, F) \otimes T([0]) \oplus F([0]) \otimes Tor^{\tilde{\Gamma}}_i(L^*, T)$$

Corollary 1.4: If $T([0]) = F([0]) = 0$ then $Tor^{\tilde{\Gamma}}_i(L^*, F \otimes T) = 0$ for every $i \geq 0$.

Theorem 1.5: There is an isomorphism for any Γ-module F and $j > 0$:

$$Tor^{\tilde{\Gamma}}_j(F \otimes L^*, \Lambda^p) \cong F[1] \otimes Tor^{\tilde{\Gamma}}_j(L^*, \Lambda^p).$$

Proof. In order to prove this theorem we need to study spectral sequence from 1.2, which in our situation looks as follows:

$${}^I E^2_{i,j} = Tor^{\widetilde{\Gamma}}_i(F, \mathbf{Tor}^I_j(L^*, \Lambda^p)) \Rightarrow Tor^{\widetilde{\Gamma}}_{i+j}(F \otimes L^*, \Lambda^p).$$

The only way to simplify this formula is to find the Γ-module $\mathbf{Tor}^I_j(L^*, \Lambda^p)$ which is equal to

$$\mathbf{Tor}^I_j(L^*, \Lambda^p)[n] \cong Tor^{\widetilde{\Gamma}}_j(L^*, \Delta_n \Lambda^p).$$

To proceed with calculations we have to analyze the following term:

$$\Delta_n \Lambda^p[m] = \Lambda^p([n] \times [m]) = \Lambda^p \circ L([n] \times [m]) = \Lambda^p(L \oplus \Gamma^1 \oplus \ldots \oplus \Gamma^1)[m]$$

with L corresponding to $0 \in [n]$ and Γ^1 to $k \in [n]$ for $1 < k \leq n$. We will denote some chosen basis of $L([n] \times [m])$ as $x_{k,s}$. To specify our preferable basis observe that there is another direct sum decomposition:

$$\Gamma^1 = \Gamma^0 \oplus L.$$

The basis of $\Gamma^1[m]$ consists of functions (base point preserving) $[1] \to [m]$ and by $x_{k,s}$ we denote the only function sending 1 to s in the kth term Γ^1 in the preceding formula. After choosing k we have inclusion of $\Gamma^0[m]$ into $\Gamma^1[m]$ sending the one and only function $[0] \to [m]$ to $x_{k,0}$. Then it is easy to see that the cokernel of this inclusion is just $L[m]$ with basis denoted as $y_{k,s}$ with $1 \leq s \leq m$. Sometimes we will need also $y_{k,0} = 0$, which fits well into all conventions. There is a section of the described above projection sending $y_{k,s} \in L[m]$ to $x_{k,s} - x_{k,0} \in \Gamma^1$.

We apply the exponential formula to the decomposition $\Gamma^1 = \Gamma^0 \oplus L$ to get :

$$\Lambda^a(\Gamma^1) = \Lambda^a(\Gamma^0 \oplus L) = \bigoplus_{t=0}^{a} \Lambda^t \circ \Gamma^0 \otimes \Lambda^{a-t} = \Lambda^1 \circ \Gamma^0 \otimes \Lambda^{a-1} \oplus \Lambda^t$$

Finally it leads to the formula:

$$\Delta_n \Lambda^p = \Lambda^p(L \oplus \Gamma^1 \oplus \ldots \oplus \Gamma^1) =$$

$$= \bigoplus_{t+t_1+t_2+\ldots+t_n=p} \{\Lambda^t \otimes (\Lambda^1 \circ \Gamma^0 \otimes \Lambda^{t_1-1} \oplus \Lambda^{t_1}) \otimes \ldots \otimes (\Lambda^1 \circ \Gamma^0 \otimes \Lambda^{t_n-1} \oplus \Lambda^{t_n})\}$$

First we will find $\mathbf{Tor}^I_j(L^*, \Lambda^p)[n]$ for $j > 0$. We have to calculate $Tor^{\widetilde{\Gamma}}_j(L^*, \Delta_n \Lambda^p)$ and we know from the formula that the Γ-module on second place is just big direct sum, so we have to identify summands that give non-trivial results. Corollary 1.4 tells us we can omit summands which are tensor products of Γ- modules with trivial value at $[0]$. It will be shown in theorem 5.1 that for $a \leq p$ and $j > 0$ the groups $Tor^{\tilde{\Gamma}}_j(L^*, \Lambda^a)$ are non-trivial for $a = p$ only, so we have to find summands which are p-th exterior powers in sense of the formula (actually all summands are p-th exterior powers but the formula shows that some of them are isomorphic to tensor products of exterior powers of smaller degrees). Finally observe that $\Lambda^1 \circ \Gamma^0 \otimes \Lambda^{t_s-1}$ is isomorphic to Λ^{t_s-1} since $\Lambda^1 \circ \Gamma^0$ is a constant functor. There is exactly one summand Λ^p corresponding to $t = p$ and n summands $\Lambda^1 \circ \Gamma^0 \otimes \Lambda^{p-1} \oplus \Lambda^p$ corresponding to $t_s = p$ for $1 \leq s \leq n$. As it was shown $\Lambda^1 \circ \Gamma^0 \otimes \Lambda^{p-1}$ can be omitted so we have $n + 1$ summands. Now we let the morphisms from Γ act on $[n]$ and we find this action exactly the same as in Γ-module Γ^1.

$$\mathbf{Tor}^I_j(L^*, \Lambda^p)[n] = Tor^{\widetilde{\Gamma}}_j(L^*, \Delta_n \Lambda^p) = \Gamma^1[n] \otimes Tor^{\tilde{\Gamma}}_j(L^*, \Lambda^p)$$

Finally we have for $j > 0$:

$$^{I}E^2_{i,j} = Tor_i^{\widetilde{\Gamma}}(F, \mathbf{Tor}_j^I(L^*, \Lambda^p)) = Tor_i^{\widetilde{\Gamma}}(F, \Gamma^1 \otimes Tor_j^{\widetilde{\Gamma}}(L^*, \Lambda^p)) \cong$$

$$\cong Tor_i^{\widetilde{\Gamma}}(F, \Gamma^1) \otimes Tor_j^{\widetilde{\Gamma}}(L^*, \Lambda^p)$$

and it is non-zero only for $i = 0$ giving one column with $F[1] \otimes Tor_j^{\widetilde{\Gamma}}(L^*, \Lambda^p)$. To complete the proof we have to calculate $\mathbf{Tor}_j^I(L^*, \Lambda^p)$ for $j = 0$:

$$\mathbf{Tor}_0^I(L^*, \Lambda^p)[n] = Tor_0^{\widetilde{\Gamma}}(L^*, \Delta_n \Lambda^p) = L^* \otimes_\Gamma \Delta_n \Lambda^p$$

$$= L^* \otimes_\Gamma \bigoplus_{t+t_1+\ldots+t_n=p} \{\Lambda^t \otimes (\Lambda^1 \circ \Gamma^0 \otimes \Lambda^{t_1-1} \oplus \Lambda^{t_1}) \otimes \ldots \otimes (\Lambda^1 \circ \Gamma^0 \otimes \Lambda^{t_n-1} \oplus \Lambda^{t_n})\}$$

First we notice that Λ^a is atomic of degree a and $\Lambda^a \otimes \Lambda^b$ has non-trivial value on $< n >$ (in the category $\tilde{\Omega}$) for $max(a, b) \leq n \leq a + b$ only. $\Lambda^1 \circ \Gamma^0$ is constant and has no impact on degree of Γ-module. L^* is atomic of degree 1 and it is easy to see that the functor $L^* \otimes_\Gamma - = cr(L^*) \otimes_\Omega -$ has non-trivial values only on summands which have non-zero first cross-effect. These are exactly tensor products of some $\Lambda^1 \circ \Gamma^0$ and some Λ^1. Corollary 1.4 tells us that we may omit summands with more than one factor Λ^1. Finally we are interested in summands of type:

$$\Lambda^1 \otimes \Lambda^1 \circ \Gamma^0 \otimes \ldots \otimes \Lambda^1 \circ \Gamma^0$$

which are obviously Γ-modules of degree 1. We group them together and by A we denote a direct sum of these. There is one factor Λ^1 and $p - 1$ factors $\Lambda^1 \circ \Gamma^0$ in each summand of A. It is obvious that factor Λ^1 might occur for $t = 1$. Then for $t_s = 1$ and $1 \leq s \leq n$ we have interesting factors in $\Lambda^1 \circ \Gamma^0 \oplus \Lambda^1$ and additionally, factors $\Lambda^1 \circ \Gamma^0 \otimes \Lambda^1$ can come from $\Lambda^1 \circ \Gamma^0 \otimes \Lambda^1 \oplus \Lambda^2$ for $t_s = 2$. We want to calculate

$$L^* \otimes_\Gamma A = cr(L^*) \otimes_\Omega cr(A) = L^*([1]) \otimes A([1]) = A([1]).$$

In the previous notation $A([1])$ has basis:

$$y_{k,1} \wedge x_{t_1,0} \wedge \ldots \wedge x_{t_{p-1},0}$$

with $0 < t_s \leq n$ and $0 \leq k \leq n$. This is the basis of $\mathbf{Tor}_0^I(L^*, \Lambda^p)[n]$ and $(\Gamma^1 \otimes \Lambda^{p-1})([n])$ as well. Direct calculation shows that the action of morphisms from Γ is exactly the same, and we have:

$$^{I}E^2_{i,0} = Tor_i^{\widetilde{\Gamma}}(F, \mathbf{Tor}_0^I(L^*, \Lambda^p)) = Tor_i^{\widetilde{\Gamma}}(F, \Gamma^1 \otimes \Lambda^{p-1}) =$$

$$= Tor_i^{\widetilde{\Gamma}}(F, \Lambda^{p-1}) \oplus Tor_i^{\widetilde{\Gamma}}(F, L \otimes \Lambda^{p-1}).$$

Both Λ^{p-1} and $L \otimes \Lambda^{p-1}$ are direct summands in the projective modules $L^{\otimes(p-1)}$ and $L^{\otimes p}$ respectively so ${}^{I}E^2_{i,0} = 0$ for $i > 0$. For $i = 0$ we have:

$$^{I}E^2_{0,0} = F \otimes_\Gamma \Lambda^{p-1} \oplus F \otimes_\Gamma (L \otimes \Lambda^{p-1}).$$

Our spectral sequence collapses and we get the desired formula for $j > 0$:

$$E^2_{0,j} = F[1] \otimes Tor^{\tilde{\Gamma}}_j(L^*, \Lambda^p) \cong Tor^{\tilde{\Gamma}}_j(F \otimes L^*, \Lambda^p).$$

This finishes the proof.

Corollary 1.6: For every $a > 0$ and $j > 0$ we have

$$Tor^{\tilde{\Gamma}}_j(L^{*\otimes a}, \Lambda^p) \cong Tor^{\tilde{\Gamma}}_j(L^*, \Lambda^p).$$

Proof. Since $a > 0$ we may use theorem 1.5 with $F = L^{*\otimes(a-1)}$. It is obvious that for $j = 0$ the corollary is not true.

II. Comparison between $\mathcal{F}$ and $\tilde{\Gamma}$.

Let us start this section from recalling the main ingredients, which allowed the authors of [FFSS] to obtain very strong computational results in the homological algebra of $\mathcal{F}$. We will denote as F any of Λ^i, S^i, D^i in the discussion below. There are four main ingredients which lead to the results of [FFSS]:

1. Direct calculation of $Ext^*_{\mathcal{F}}(Id, F)$ (obtained in [FLS]).

2. Two sided adjointness of functors $\pi : \mathcal{V}_p \times \mathcal{V}_p \to \mathcal{V}_p$ and $\Delta : \mathcal{V}_p \to \mathcal{V}_p \times \mathcal{V}_p$ where the first functor is given by sum and the second by the diagonal map.

3. Exponentiality of F:

$$F^n(V \oplus W) = \oplus^n_{i=0} F^i(V) \oplus F^{n-i}(W)$$

4. The fact (roughly speaking) that in some cases one can go with the action of the symmetric group through the Ext-sign:

$$(Ext^*_{\mathcal{F}}(Id^{\otimes n}, F))^{\Sigma_n} = Ext^*_{\mathcal{F}}((Id^{\otimes n})^{\Sigma_n}, F)$$

One explanation is necessary at this point. An expert can say, that there was the fifth ingredient; the use of the category of *strict* polynomial functors instead of $\mathcal{F}$. But, first of all, this was needed for achieving 4. Hence 4 is the goal, whatever method one applies to obtain it. Secondly, *strict* polynomial functors came from algebraic geometry and it is hard to imagine this direction in studying $\tilde{\Gamma}$. This explains why *strict* polynomials functors are skipped from our considerations.

Let us have a closer look at the steps 1-4 from the point of view of category $\tilde{\Gamma}$. Obviously for $Tor^{\tilde{\Gamma}}$-calculations we exchange Id by L^*. The calculation of $Tor_*^{\tilde{\Gamma}}(L^*, F)$ is done in [B1] so we can proceed further, to the ingredient 2. And this is the crucial step for the rest of the program and the crucial difference between two categories under consideration. Let us call $bi-\mathcal{F}$ the category of functors $\mathcal{V}_p \times \mathcal{V}_p \to Vect_{F_p}$. We have the following formulas in $\mathcal{F}$ (compare [FFSS, formula 1.7.1]), coming directly from 2 above:

Theorem 2.1: Let $T \in \mathcal{F}$ and $S \in bi-\mathcal{F}$. Then

$$Ext^*_{\mathcal{F}}(T \circ \Delta, S) = Ext^*_{bi-\mathcal{F}}(T, S \circ \pi)$$

$$Ext^*_{\mathcal{F}}(T, S \circ \Delta) = Ext^*_{bi-\mathcal{F}}(T \circ \pi, S)$$

The first equality leads to a quick calculation of $Ext^*_{\mathcal{F}}(Id^{\otimes n}, F)$ for the exponential F because in $bi-\mathcal{F}$ we have the Kunneth formula. Let us check whether point 2 is valid in Γ. We have the maps: diagonal one has the obvious definition and π takes a pair of pointed sets X and Y to their wedge. But this two functors satisfy only one adjointness formula:

Lemma 2.2: $Hom_{\Gamma\times\Gamma}((Y, Z), (X, X)) = Hom_{\Gamma}(Y \vee Z, X)$.

which gives us

Lemma 2.3: $Ext^*_{\tilde{\Gamma}}(E, F \circ \Delta) = Ext^*_{bi-\tilde{\Gamma}}(E \circ \pi, F)$

This allows to treat the situations when one has to deal with Ext-groups from an exponential functor to a tensor product of functors. Unfortunately, neither the second adjointness nor the first part of 2.1 is true in $\tilde{\Gamma}$. Examples of this phenomenon in the language of Tor-groups were shown in the previous section, perhaps the easiest one is given by the formula from 1.6:

$$Tor_*^{\tilde{\Gamma}}(L^* \otimes L^*, \Lambda^p) = Tor_*^{\tilde{\Gamma}}(L^*, \Lambda^p)$$

It is easy to calculate that in the Ext-world one has

$$Ext^*_{\tilde{\Gamma}^{op}}(L^* \otimes L^*, \Lambda^p) = Ext^*_{\tilde{\Gamma}^{op}}(L^*, \Lambda^p)$$

III. Preliminary calculations in degree 0.

As one can imagine knowing published papers on homological algebra in $\mathcal{F}$ and $\tilde{\Gamma}$, in our calculations we will use Koszul and de Rham sequences which relate exterior and symmetric powers. The middle terms in this sequences are given by tensor products of the same type of functors. This, very simple section, shows that at least categorical tensor products of such functors are computable. Of course we will treat mostly the cases which are needed in further calculations. Remember that since now up to the end our functors are of degree $\leq p$.

First we want to list few cases when tensor products of some interesting modules are trivial, or easy to describe. First lemma is just an easy consequence of the definition. We work mostly in the category $\tilde{\Omega}$, it was justified in section I.

Lemma 3.1:

$$cr(\Lambda^a) \otimes_\Omega cr(\Lambda^b) = \begin{cases} 0 & \text{for } a \neq b, \\ F_p & \text{otherwise} \end{cases}$$

Proof is obvious. Next lemma generalizes this computation using the same observation that tensor product of Ω-modules concentrated in different dimensions is trivial.

Lemma 3.2: Let $0 \leq i \leq a$, then:

$$cr(\Lambda^i \otimes S^{a-i}) \otimes_\Omega cr(\Lambda^j \otimes S^{b-j}) = \begin{cases} 0 & \text{for } a < j, \\ 0 & \text{for } b < i \end{cases}$$

There is still something to say in other cases.

Lemma 3.3: Let $0 \leq i \leq a$, then:

$$cr(\Lambda^a) \otimes_\Omega cr(\Lambda^{a-i} \otimes S^i) = \begin{cases} 0 & \text{for } i \geq 2, \\ F_p & \text{for } i = 1, \\ F_p & \text{for } i = 0 \end{cases}$$

$$cr(\Lambda^{a-i} \otimes S^i) \otimes_\Omega cr(\Lambda^a) = \begin{cases} 0 & \text{for } i \geq 2, \\ F_p & \text{for } i = 1, \\ F_p & \text{for } i = 0 \end{cases}$$

Proof. These formulas require short comment, but first we need some notation. Our functors are of degree a so it is enough to evaluate them on the set of a elements. Let $x_1, ..., x_a$ denote the basis of $L([a])$ and $x_1^*, ..., x_a^*$ the dual basis of $L^*([a])$. The first formula is easier to show. Passing to cross-effects we see that $cr(\Lambda^a)$ is atomic and generated by $x_1^* \wedge \cdots \wedge x_a^*$, while $cr(\Lambda^{a-i} \otimes S^i)$ in dimension a has generators of form $x_{r_1} \wedge \cdots \wedge x_{r_{a-i}} \otimes x_{r_{a-i+1}} \cdots x_{r_a}$. We get tensor product in Ω-modules tensoring these as vector spaces and dividing the result by relations coming from the action of Σ_a. For $i \geq 2$ we have:

$$x_1^* \wedge \cdots \wedge x_a^* \otimes x_{r_1} \wedge \cdots \wedge x_{r_{a-i}} \otimes x_{r_{a-i+1}} \cdots x_{r_a} =$$

$$= x_1^* \wedge \cdots \wedge x_a^* \otimes (r_{a-1}, r_a) \cdot [x_{r_1} \wedge \cdots \wedge x_{r_{a-i}} \otimes x_{r_{a-i+1}} \cdots x_{r_a}] =$$

$$= -x_1^* \wedge \cdots \wedge x_a^* \otimes x_{r_1} \wedge \cdots \wedge x_{r_{a-i}} \otimes x_{r_{a-i+1}} \cdots x_{r_a}$$

where (r_{a-1}, r_a) denote the transposition of two numbers. So we see that we have to divide by $2x_1^* \wedge \cdots \wedge x_a^* \otimes x_{r_1} \wedge \cdots \wedge x_{r_{a-i}} \otimes x_{r_{a-i+1}} \cdots x_{r_a}$. By assumption characteristic of our field is different from 2, so this way we get rid of all generators. When $i = 1$ similar calculations show us that:

$$x_1^* \wedge \cdots \wedge x_a^* \otimes x_{r_1} \wedge \cdots \wedge x_{r_{a-1}} \otimes x_{r_a} =$$

$$= x_1^* \wedge \cdots \wedge x_a^* \otimes (r_{a-1}, r_a) \cdot [x_{r_1} \wedge \cdots \wedge x_{r_a} \otimes x_{r_{a-1}}] =$$
$$= -x_1^* \wedge \cdots \wedge x_a^* \otimes x_{r_1} \wedge \cdots \wedge x_{r_a} \otimes x_{r_{a-1}}$$

and this way we will get only one generator. Case of $i = 0$ is easy as it is tensor product over Σ_p of two sign representations.

When we want to calculate $(\Lambda^{a-i} \otimes S^i) \otimes_\Gamma \Lambda^a$ we have to be a little bit more careful. When $i \neq 1$ arguments are exactly the same, but for $i = 1$ we have new relations coming from smaller dimensions. But it is clear that this chosen generator $x_1^* \wedge \cdots \wedge x_{a-1}^* \otimes x_a^* \otimes x_1 \wedge \cdots \wedge x_a$ is not hit by them.

⋆

When $0 < a < p$ it is well known that S^a and Λ^a are direct summands in $L^{\otimes a}$. In consequence, $\Lambda^{p-i} \otimes S^i$ is direct summand in $L^{\otimes p}$ for $0 < i < p$. Next lemma generalizes previous results and will be most commonly used in our further considerations.

Lemma 3.4: Let F be a Γ-module with $cr_a(F) = 0$ and G be some direct summand in $L^{\otimes a}$. Then

$$F \otimes_\Gamma G = 0$$

Proof. As usual we pass to the cross-effects and get:

$$F \otimes_\Gamma G \subseteq F \otimes_\Gamma L^{\otimes a} \cong cr(F) \otimes_\Omega cr(L^{\otimes a}) \cong cr(F) \otimes_\Omega \Omega^a \cong cr_a(F) = 0$$

where $\Omega^a = F_p[Hom_\Omega(< a >, -)]$. It is a direct calculation to show that $\Omega^a = cr(L^{\otimes a})$ (compare [P1, page 160]). Hence the last "$\cong$" above comes from the Yoneda lemma (for the tensor product).

Lemma 3.5: Let $0 \leq i \leq p$, then:

$$cr(S^p) \otimes_\Omega cr(\Lambda^{p-i} \otimes S^i) = \begin{cases} 0 & \text{for } i \leq p-2, \\ F_p & \text{for } i = p-1, \\ F_p & \text{for } i = p \end{cases}$$

Proof. Let us start from the case $i = p$. By the right exactness of the tensor product we know that we have an epimorphism

$$F_p = S^p \otimes_\Gamma L^{\otimes p} \to S^p \otimes_\Gamma S^p$$

and it is easy to see that $x_1^* ... x_p^* \otimes x_1 ... x_p$ generates a nontrivial 1 dimensional summand in the target.

Since $\Lambda^{p-i} \otimes S^i$ is a direct summand in $L^{\otimes p}$ for $i \geq 1$ we have:

$$S^p \otimes_\Gamma \Lambda^{p-i} \otimes S^i \subseteq S^p \otimes_\Gamma L^{\otimes p} \cong cr(S^p) \otimes_\Omega cr(L^{\otimes p}) \cong cr(S^p) \otimes_\Omega \Omega^p \cong F_p$$

The generator of $cr(S^p) \otimes_\Omega cr(L^{\otimes p})$ is $x_1^* \cdots x_p^* \otimes x_1 \otimes \cdots \otimes x_p$. It is easy to observe that $x_1^* \cdots x_p^* \otimes x_1 \wedge \cdots \wedge x_{p-i} \otimes x_{p-i+1} \cdots x_p$ generates $S^p \otimes_\Gamma \Lambda^{p-i} \otimes S^i$. Hence we have to check only when $x_1^* \cdots x_p^* \otimes x_1 \wedge \cdots \wedge x_{p-i} \otimes x_{p-i+1} \cdots x_p$ is nontrivial. The same kind of arguments as in the proof of the previous lemma show that this element is zero when $i \leq p-2$ and give one generator when $i = p-1$. This last case can be also easily seen in the spirit of section V, via tensoring K_*^p with S^p.

IV. Preliminary calculations in higher degrees.

This short section is devoted to the preliminary calculations of the higher Tor-functors. Most of the results here are very simple but we have to state them for further reference. In all formulas of the type $Tor_i^{\tilde{\Gamma}}(-,-)$ we will assume that $i > 0$. We will work in the categories $\tilde{\Gamma}$ and $\tilde{\Omega}$ which are equivalent. We start from considering functors of degree smaller than p. The first two lemmas are obvious but let us recall these facts again:

Lemma 4.1: If $0 < a < p$ then for any functor F:

$$Tor_i^{\tilde{\Gamma}}(F, \Lambda^a) = 0$$

$$Tor_i^{\tilde{\Gamma}}(F, S^a) = 0$$

Proof. It is obvious because under our assumptions both Λ^a and S^a are projective as direct summands of a projective object $L^{\otimes a}$.

Lemma 4.2: If $1 < b \leq p$ and $0 < j < b$ then for any functor F:

$$Tor_i^{\tilde{\Gamma}}(F, \Lambda^j \otimes S^{b-j}) = 0$$

Proof. As previously, we use the fact that for $b \leq p$ and $0 < j < b$, $\Lambda^j \otimes S^{b-j}$ is direct summand in a projective object $L^{\otimes b}$.

The following lemma will be crucial to proceed with calculations when one of our Γ-modules is equal to the p-th symmetric power.

Lemma 4.3: For every Γ-module G group $Tor_i^{\tilde{\Gamma}}(L^*, G)$ is direct summand in $Tor_i^{\tilde{\Gamma}}(S^p, G)$.

Proof.

Observe that L^* is direct summand in $S^p \circ L^*$. It might be easily checked in the category $\tilde{\Omega}$. One can see that there is quotient map $S^p \circ L^* \to L^*$ obtained by moding out all cross effects above degree 1. The Frobenius morphism gives us the splitting. Hence

$$L^* \oplus X \cong S^p$$

and

$$Tor_i^{\tilde{\Gamma}}(S^p, G) \cong Tor_i^{\tilde{\Gamma}}(L^*, G) \oplus Tor_i^{\tilde{\Gamma}}(X, G).$$

Lemma 4.4: If $1 < a < p$ then:

$$Tor_i^{\tilde{\Gamma}}(\Lambda^a, \Lambda^p) = 0$$

Proof. Once again we use the formula:

$$\Lambda^a \oplus F_a \cong L^{*\otimes a}$$

for some functor F_a to get:

$$Tor_i^{\tilde{\Gamma}}(\Lambda^a, \Lambda^p) \oplus Tor_i^{\tilde{\Gamma}}(F_a, \Lambda^p) \cong Tor_i^{\tilde{\Gamma}}(L^{*\otimes a}, \Lambda^p) \cong Tor_i^{\tilde{\Gamma}}(L^*, \Lambda^p)$$

where the last equivalence comes from 1.6. Moreover we know that the last equivalence comes from the quotient map to the first cross-effect $L^{*\otimes a} \to L^*$. We know also that the first cross-effects of F_a and $L^{*\otimes a}$ are naturally the same. This gives us the desired formula. We will get the same result in 5.1 in another way.

V. Main calculations.

Let us start from recalling that groups $Tor_i^{\tilde{\Gamma}}(L^*, \Lambda^b)$ were calculated in [B1] with the crucial help of Koszul and de Rham complexes. We recall and extend these computations in Theorem 5.1 below.

We should start from recalling necessary notions. Koszul sequence $\mathbf{K}^n$ is a sequence of Γ-modules $\mathbf{K}^n_j = \Lambda^j \otimes S^{n-j}$:

$$0 \to \Lambda^n \to \Lambda^{n-1} \otimes S^1 \to ... \to \Lambda^1 \otimes S^{n-1} \to S^n \to 0$$

which is exact for any n. Similarly we have de Rham sequence $\mathbf{R}^n_j = \Lambda^{n-j} \otimes S^j$:

$$0 \leftarrow \Lambda^n \leftarrow \Lambda^{n-1} \otimes S^1 \leftarrow ... \leftarrow \Lambda^1 \otimes S^{n-1} \leftarrow S^n \leftarrow 0.$$

which is exact for n relatively prime to p and $H_{i+pk-k}(\mathbf{R}^{pk}) = \mathbf{R}^k_i$.

Theorem 5.1: Let $1 \leq a < p$. Then

$$Tor_j^{\tilde{\Gamma}}(\Lambda^a, \Lambda^p) = \begin{cases} F_p & \text{for a=1, } j = (2s+1)(p-1)+1, \ s \geq 0 \\ F_p & \text{for a=1, } j = (2s+1)(p-1), \ s \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We will divide our arguments in two steps.

Step 1. Let us have a look at the hyperhomology spectral sequences (two of them) with coefficients in the Koszul complex.

$${}^{I}E^2_{i,j} = Tor_i^{\tilde{\Gamma}}(\Lambda^a, H_j(\mathbf{K}^p)) \Rightarrow HTor_{i+j}(\Lambda^a, \mathbf{K}^p)$$

and

$${}^{II}E^2_{i,j} = H_i(Tor_j^{\tilde{\Gamma}}(\Lambda^a, \mathbf{K}^p)) \Rightarrow HTor_{i+j}(\Lambda^a, \mathbf{K}^p)$$

The first converges to zero since Koszul complex is acyclic, so does the second. On the other hand second spectral sequence ${}^{II}E^2_{i,j}$ has possibly non-trivial terms in columns $i = 0, p$. Indeed, $Tor^{\tilde{\Gamma}}_j(\Lambda^a, S^b \otimes \Lambda^{p-b}) = 0$ for $a < p$ as it was shown before (compare lemmas 3.4 and 4.2). The only nontrivial differential on ${}^{II}E^p_{i,j}$ induces an isomorphism for $j \geq 0$:

$$d^p_{p,j} : Tor^{\tilde{\Gamma}}_j(\Lambda^a, \Lambda^p) \cong Tor^{\tilde{\Gamma}}_{j+p-1}(\Lambda^a, S^p) \qquad [5.1.1]$$

Obviously, for $p - 2 \geq j \geq 0$ we have:

$$Tor^{\tilde{\Gamma}}_j(\Lambda^a, S^p) = 0$$

by dimension reason.

Step 2. Now we turn our attention to the hyperhomology spectral sequences with coefficients in the de Rham complex. Situation is now slightly different, since de Rham complex is not acyclic. In fact it has $H_{p-1}(\mathbf{R}^p) = H_p(\mathbf{R}^p) = L$, so we have possibly two nontrivial columns in ${}^{I}E^2_{i,j}$ for $i = (p-1)$ and $i = p$. First we consider case $a \neq 1$. We know that $Tor^{\tilde{\Gamma}}_*(\Lambda^a, L) = 0$ so first spectral sequence converges to 0. Hence second does the same. The second spectral sequence has only two nontrivial columns for exactly the same reason as in the case of $\mathbf{K}^p$, but this time $Tor^{\tilde{\Gamma}}_j(\Lambda^a, S^p)$ stands in p^{th} column and $Tor^{\tilde{\Gamma}}_j(\Lambda^a, \Lambda^p)$ in ${}^{II}E^2_{0,j}$. Hence we get

$$d^p_{p,j} : Tor^{\tilde{\Gamma}}_j(\Lambda^a, S^p) \cong Tor^{\tilde{\Gamma}}_{j+p-1}(\Lambda^a, \Lambda^p) \qquad [5.1.2]$$

and for $p - 2 \geq j \geq 0$ we have:

$$Tor^{\tilde{\Gamma}}_j(\Lambda^a, \Lambda^p) = 0$$

Comparing Koszul and de Rham calculations we get:

$$Tor^{\tilde{\Gamma}}_j(\Lambda^a, \Lambda^p) \cong Tor^{\tilde{\Gamma}}_{j+2(p-1)}(\Lambda^a, \Lambda^p) = 0$$

Case $a = 1$ was calculated in [B1], we give the proof for the sake of completeness. First step is similar with necessary changes. We get an isomorphism for $j \geq 0$:

$$Tor^{\tilde{\Gamma}}_j(L^*, \Lambda^p) \cong Tor^{\tilde{\Gamma}}_{j+p-1}(L^*, S^p) \qquad [5.1.3]$$

and $Tor^{\tilde{\Gamma}}_j(L^*, S^p) = 0$ for $j < p - 1$. The first spectral sequence with coefficients in the de Rham complex reduces to only two groups by the formula for the homology of $\mathbf{R}^p$, so the spectral sequence collapses and we have:

$${}^{I}E^{\infty}_{p-1,0} = {}^{I}E^2_{p-1,0} = Tor^{\tilde{\Gamma}}_0(L^*, L) = F_p, \quad \text{and} \quad {}^{I}E^{\infty}_{p,0} = {}^{I}E^2_{p,0} = Tor^{\tilde{\Gamma}}_0(L^*, L) = F_p$$

As previously the second spectral sequence has only two nontrivial columns and again $Tor^{\tilde{\Gamma}}_j(\Lambda^a, S^p)$ stands in p^{th} column and $Tor^{\tilde{\Gamma}}_j(\Lambda^a, \Lambda^p)$ in ${}^{II}E^2_{0,j}$. We have to examine carefully an exact sequence induced by the differential on ${}^{II}E^p_{i,j}$:

$$d^p_{p,j} : Tor^{\tilde{\Gamma}}_j(L^*, S^p) \to Tor^{\tilde{\Gamma}}_{j+p-1}(L^*, \Lambda^p)$$

From the very definition we have:

$$0 \to^{II} E^{\infty}_{p,j} \to^{II} E^{p}_{p,j} = Tor^{\tilde{\Gamma}}_{j}(L^*, S^p) \to Tor^{\tilde{\Gamma}}_{j+p-1}(L^*, \Lambda^p) =^{II} E^{p}_{0,j+p-1} \to^{II} E^{\infty}_{0,j+p-1} \to 0.$$

We know that this spectral sequence converges to trivial groups in all dimensions except the following two terms:

$$HTor_{p-1} = {}^{I}E^{\infty}_{p-1,0} = F_p \quad \text{and} \quad HTor_p = {}^{I}E^{\infty}_{p,0} = F_p$$

These two vector spaces must be found among terms ${}^{II}E^{\infty}_{i,j}$ with $i + j = (p-1)$ and $i + j = p$ in the second spectral sequence. We know that $Tor^{\tilde{\Gamma}}_{i}(L^*, S^p) = 0$ for $i < p-1$ so we get immediately that:

$$Tor^{\tilde{\Gamma}}_{p-1}(L^*, \Lambda^p) = F_p$$

$$Tor^{\tilde{\Gamma}}_{p}(L^*, \Lambda^p) = F_p.$$

For $j > 1$ we have an isomorphism:

$$Tor^{\tilde{\Gamma}}_{j}(L^*, S^p) \cong Tor^{\tilde{\Gamma}}_{j+p-1}(L^*, \Lambda^p)$$

and $Tor^{\tilde{\Gamma}}_{j}(L^*, \Lambda^p) = 0$ for $j < p-1$. These formulas and 5.1.3 give us isomorphism for $j > 1$:

$$Tor^{\tilde{\Gamma}}_{j}(L^*, \Lambda^p) \cong Tor^{\tilde{\Gamma}}_{j+2(p-1)}(L^*, \Lambda^p).$$

This completes the proof.

From 5.1 we have an obvious corollary

Corollary 5.2: For $1 \le a < p$

$$Tor^{\tilde{\Gamma}}_{j}(\Lambda^a, S^p) = \begin{cases} F_p & \text{for a=1, } j = (2s+2)(p-1)+1, s \ge 0 \\ F_p & \text{for a=1, } j = (2s+2)(p-1), s \ge 0 \\ 0 & \text{otherwise.} \end{cases}$$

Now we can move towards more serious calculations. First we have:

Theorem 5.3:

$$Tor^{\tilde{\Gamma}}_{j}(\Lambda^p, \Lambda^p) = \begin{cases} F_p & \text{for } j = (2s+2)(p-1)-1, s \ge 0 \\ F_p & \text{for } j = 2s(p-1), s \ge 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We will proceed as in 5.1. But now first step needs some comments. First hyperhomology spectral sequence with coefficients in Koszul complex converges to zero and so does the second as previously. Now we have to look more carefully on first stage of the second one ${}^{II}E^{1}_{i,j}$, because nonzero terms may appear not only in columns $i = 0, p$, but in the row $j = 0$ either. Indeed, $Tor^{\tilde{\Gamma}}_{j}(\Lambda^p, S^b \otimes \Lambda^{p-b}) = 0$ for $p > b > 0$ and $j > 0$ since

$S^b \otimes \Lambda^{p-b}$ is projective, and when $j = 0$ we know that $Tor_0^{\tilde{\Gamma}}(\Lambda^p, S^b \otimes \Lambda^{p-b}) = \Lambda^p \otimes_\Gamma (S^b \otimes \Lambda^{p-b})$ is zero when $b > 1$, but there is still one nonzero term which is $\Lambda^p \otimes_\Gamma (S^1 \otimes \Lambda^{p-1})$ standing in $E^1_{p-1,0}$. In consequence there is one possibly nontrivial differential on the first stage:

$$d^1_{p,0} : E^1_{p,0} = \Lambda^p \otimes_\Gamma \Lambda^p \to \Lambda^p \otimes_\Gamma (S^1 \otimes \Lambda^{p-1}) = E^1_{p-1,0}$$

Direct calculation below shows that it is zero.

It is easy to see that $\Lambda^p \otimes_\Gamma (S^1 \otimes \Lambda^{p-1})$ is one dimensional. It is generated by $x_1^* \wedge \cdots \wedge x_p^* \otimes x_1 \otimes x_2 \wedge \cdots \wedge x_p$, in the notation of 3.3. Indeed, Λ^p has one generator, and $(S^1 \otimes \Lambda^{p-1})$ has p generators $x_j \otimes x_1 \wedge \cdots \wedge x_{j-1} \wedge x_{j+1} \wedge \cdots \wedge x_p$. In the tensor product over Ω we have p-dimensional vector space generated by the tensor product over F_p of these, and we divide by relations coming from the action of Σ_p:

$$x_1^* \wedge \cdots \wedge x_p^* \otimes x_j \otimes x_1 \wedge \cdots \wedge x_{j-1} \wedge x_{j+1} \wedge \cdots \wedge x_p =$$

$$= -(x_1^* \wedge \cdots \wedge x_p^*)(1j) \otimes x_j \otimes x_1 \wedge \cdots \wedge x_{j-1} \wedge x_{j+1} \wedge \cdots \wedge x_p =$$

$$= -x_1^* \wedge \cdots \wedge x_p^* \otimes (1j)(x_j \otimes x_1 \wedge \cdots \wedge x_{j-1} \wedge x_{j+1} \wedge \cdots \wedge x_p) =$$

$$= -x_1^* \wedge \cdots \wedge x_p^* \otimes x_1 \otimes x_j \wedge \cdots \wedge x_{j-1} \wedge x_{j+1} \wedge \cdots \wedge x_p =$$

$$= -x_1^* \wedge \cdots \wedge x_p^* \otimes (-1)^{j-2} x_1 \otimes x_2 \wedge \cdots \wedge x_{j-1} \wedge x_j \wedge x_{j+1} \wedge \cdots \wedge x_p =$$

$$= (-1)^{j-1} x_1^* \wedge \cdots \wedge x_p^* \otimes x_1 \otimes x_2 \wedge \cdots \wedge x_p$$

Recall that $(1j)$ denotes here the transposition interchanging 1 and j. Our differential is now:

$$d^1_{p,0}(x_1^* \wedge \cdots \wedge x_p^* \otimes x_1 \wedge \cdots \wedge x_p) =$$

$$= \sum_{j=1}^{p} (-1)^{j-1} x_1^* \wedge \cdots \wedge x_p^* \otimes x_j \otimes x_1 \wedge \cdots \wedge x_{j-1} \wedge x_{j+1} \wedge \cdots \wedge x_p =$$

$$= \sum_{j=1}^{p} (-1)^{2(j-1)} x_1^* \wedge \cdots \wedge x_p^* \otimes x_1 \otimes x_2 \wedge \cdots \wedge x_p =$$

$$= \sum_{j=1}^{p} x_1^* \wedge \cdots \wedge x_p^* \otimes x_1 \otimes x_2 \wedge \cdots \wedge x_p =$$

$$= p x_1^* \wedge \cdots \wedge x_p^* \otimes x_1 \otimes x_2 \wedge \cdots \wedge x_p = 0$$

In consequence, apart from two possibly nontrivial columns, we have one more nontrivial term on the second stage $E^2_{p-1,0} = \Lambda^p \otimes_\Gamma (S^1 \otimes \Lambda^{p-1}) = F_p$. Now nothing happens until we reach $(p-1)^{st}$ stage of our spectral sequence where we have one non-trivial differential:

$$d^{p-1}_{p-1,0} : E^{p-1}_{p-1,0} = E^2_{p-1,0} \to Tor^{\tilde{\Gamma}}_{p-2}(\Lambda^p, S^p) = E^2_{0,p-2}.$$

On the p^{th} stage we have isomorphisms for $j \geq 0$:

$$d^p_{p,j} : Tor^{\tilde{\Gamma}}_j(\Lambda^p, \Lambda^p) \cong Tor^{\tilde{\Gamma}}_{j+p-1}(\Lambda^p, S^p), \qquad [5.3.1]$$

Observe that we still have for $0 \leq i < p-2$:

$$Tor_i^{\tilde{\Gamma}}(\Lambda^p, S^p) = 0. \qquad [5.3.2]$$

Finally our spectral sequence converges to zero so:

$$E^p_{0,p-2} = E^\infty_{0,p-2} = 0$$

and the differential $d^{p-1}_{p-1,0}$ must have been an epimorphism. Similarly $E^p_{p-1,0}$ survives to infinity so it must be trivial since spectral sequence converges to 0. That means $d^{p-1}_{p-1,0}$ must have been monomorphism and an isomorphism.

$E^2_{p-1,0} = \Lambda^p \otimes_\Gamma (S^1 \otimes \Lambda^{p-1}) = F_p$ so we have:

$$Tor^{\tilde{\Gamma}}_{p-2}(\Lambda^p, S^p) \cong F_p, \qquad [5.3.3]$$

From 5.3.1 we get:

$$Tor^{\tilde{\Gamma}}_{p-1}(\Lambda^p, S^p) \cong Tor^{\tilde{\Gamma}}_0(\Lambda^p, \Lambda^p) = F_p, \qquad [5.3.4]$$

This completes first step of our calculations. Now we look at the hyperhomology with coefficients in the de Rham complex. First spectral sequence converges to zero as in previous theorem since the only possibly non-trivial terms on ${}^I E^2_{p,q}$ are:

$$E^\infty_{p-1,0} = E^2_{p-1,0} = Tor^{\tilde{\Gamma}}_0(\Lambda^p, L) = 0, \text{ and } E^\infty_{p,0} = E^2_{p,0} = Tor^{\tilde{\Gamma}}_0(\Lambda^p, L) = 0.$$

On ${}^{II}E^1_{i,j}$ we have two nonzero columns $i = 0, p$ and one additional term on the row $j = 0$. It is just as previously $\Lambda^p \otimes_\Gamma (S^1 \otimes \Lambda^{p-1})$ standing in $E^1_{1,0}$ (this time $Tor^{\tilde{\Gamma}}_*(\Lambda^p, S^p)$ stand in p^{th} column while $Tor^{\tilde{\Gamma}}_*(\Lambda^p, \Lambda^p)$ in the column number 0). In consequence there is one possibly nontrivial differential on the first stage:

$$d^1_{1,0} : E^1_{1,0} = \Lambda^p \otimes_\Gamma (S^1 \otimes \Lambda^{p-1}) \to \Lambda^p \otimes_\Gamma \Lambda^p = E^1_{0,0}$$

But this time we know that it is an epimorphism on the one dimensional vector space since tensor product is right exact and $Tor^{\tilde{\Gamma}}_0(\Lambda^p, \Lambda^p) = F_p$. In fact it is even isomorphism as any epimorphism from one dimensional space on F_p.

Now nothing happens until we reach ${}^{II}E^p_{i,j}$ where we have series of isomorphisms for $j \geq 0$:

$$d^p_{p,j} : Tor^{\tilde{\Gamma}}_j(\Lambda^p, S^p) \cong Tor^{\tilde{\Gamma}}_{j+p-1}(\Lambda^p, \Lambda^p), \qquad [5.3.5]$$

and additionally for $p-2 \geq i > 0$:

$$Tor^{\tilde{\Gamma}}_i(\Lambda^p, \Lambda^p) = 0. \qquad [5.3.6]$$

Formulas [5.3.1]-[5.3.6] give us desired result.

Corollary 5.4:

$$Tor_j^{\tilde{\Gamma}}(\Lambda^p, S^p) = \begin{cases} F_p & \text{for } j = (2s+1)(p-1) - 1 \ , s \geq 0 \\ F_p & \text{for } j = (2s+1)(p-1) \ , s \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Remark 5.5: It is easy to see that $Tor_j^{\tilde{\Omega}}(\Lambda^p, \Lambda^p) = H_j(\Sigma_p, F_p)$. Hence theorem 5.3 can be viewed as calculation of the mod-p homology of the symmetric group Σ_p. This is the point which we are going to push forward in our next papers and give a new way of calculating mod-p homology of Σ_n for any n.

To finish our calculations we need formulas for $Tor_j^{\tilde{\Gamma}}(S^p, \Lambda^p)$ and $Tor_j^{\tilde{\Gamma}}(S^p, S^p)$. We will get them in the same spirit as previously, analyzing spectral sequences for $HTor(S^p, X)$ where X will be Koszul or de Rham sequence.

Theorem 5.6: We have following formulas

$$Tor_j^{\tilde{\Gamma}}(S^p, \Lambda^p) = \begin{cases} F_p & \text{for } j = (2s+1)(p-1) + c \ where \ c = -1, 1, s \geq 0 \\ F_p \oplus F_p & \text{for } j = (2s+1)(p-1), s \geq 0 \\ 0 & \text{otherwise} \end{cases} .$$

and

$$Tor_j^{\tilde{\Gamma}}(S^p, S^p) = \begin{cases} F_p & \text{for } j = (2s+2)(p-1) + c \ where \ c = -1, 1, s \geq 0 \\ F_p & \text{for } j = 0 \\ F_p \oplus F_p & \text{for } j = (2s+2)(p-1), s \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The proof is very similar to the proofs of 5.1 and 5.3. We wrote them with all details so here we will give details only at places where one needs some additional argument. The spectral sequence for $HTor$-groups with coefficients in $\mathbf{K}^p$ converges to 0. It has two nontrivial columns and one additional F_p at $(1,0)$. This latter F_p kills $Tor_0^{\tilde{\Gamma}}(S^p, S^p) = F_p$ on the first stage. The fact that this sequence converges to 0 gives us the appropriate shift in dimension between $Tor_j^{\tilde{\Gamma}}(S^p, S^p)$ and $Tor_j^{\tilde{\Gamma}}(S^p, \Lambda^p)$.

The first spectral sequence with coefficients in $\mathbf{R}^p$ has only two nontrivial groups on the second table ${}^IE^2_{p-1,0} = {}^I E^2_{p,0} = F_p$. Hence it converges to two F_p's in dimensions $p-1$ and p, and so does the second spectral sequence. The latter one has possibly two nontrivial columns number 0 and p. Additionally we have one more nontrivial group:

$${}^{II}E^1_{p-1,0} = F_p$$

as we observed previously. One can check by hands that $d^1_{p,p-1} = 0$. By the general diagram chasing one can get formulas of Theorem 5.6 if one proves that both F_p's standing at ${}^{II}E^1_{p-1,0}$ and ${}^{II}E^1_{p,0}$ do not survive to ${}^{II}E^{\infty}_{*,*}$. And for this we need some additional argument.

Observe that we have a map of complexes $\phi : X \to \mathbf{R}^p$ where X has only two nontrivial terms equal to L in dimension $p-1$ and p. The map $X_p \to X_{p-1}$ is trivial and ϕ is obviously a quasi-isomorphism. Hence ϕ induces an isomorphism

$$HTor_*(S^p, X) \to HTor_*(S^p, \mathbf{R}^p)$$

On the other hand, direct inspection shows that $\phi_{p-1} : L \to \Lambda^1 \otimes S^{p-1}$ and $\phi_p : L \to S^p$ induce trivial maps on $Tor_0^{\tilde{\Gamma}}(S^p, .)$-groups. That means the composition:

$$Tor_0^{\tilde{\Gamma}}(S^p, L) = HTor_j(S^p, X) = HTor_j(S^p, \mathbf{R}^p) \to^{II} E^{\infty}_{j,0}$$

with the last map quotient, is trivial for $j = p, p-1$. This tells us immediately that these two nontrivial F_p summands of $HTor_*(S^p, \mathbf{R}^p)$ should appear on the 0 column of the second spectral sequence for $HTor_*(S^p, \mathbf{R}^p)$. But this forces our two differentials

$$d^{p-1}_{p-1,0} :^{II} E^{p-1}_{p-1,0} \to^{II} E^{p-1}_{0,p-2}$$

and

$$d^p_{p,0} :^{II} E^p_{p,0} \to^{II} E^p_{0,p-1}$$

to be nontrivial.

Remark 5.7: As was observed in 4.3, the functor L^* splits as a direct summand from $S^p \circ L^*$. Hence we have the corresponding splitting on Tor-groups:

$$Tor_j^{\tilde{\Gamma}}(S^p, S^p) = U_j \oplus Tor_j^{\tilde{\Gamma}}(L^*, S^p)$$

for a certain graded vector space U_j. It is easy to check from our computations that $U_j = Tor_j^{\tilde{\Gamma}}(\Lambda^p, \Lambda^p)$. Hence one can say that the graded group $Tor_j^{\tilde{\Gamma}}(S^p, S^p)$ consists of the homology of $H_j(\Sigma_p, F_p)$ (corresponding to $Tor_j^{\tilde{\Gamma}}(\Lambda^p, \Lambda^p)$) and the part of the Steenrod algebra corresponding to $Tor_j^{\tilde{\Gamma}}(L^*, S^p)$ (compare [B1]).

We proceed now with some less interesting cases, which will be useful in future calculations for the functors of higher degree. In Corollary 1.6 we proved that for $s > 0$ we have:

$$Tor_s^{\tilde{\Gamma}}(L^{*\otimes a}, \Lambda^p) \cong Tor_s^{\tilde{\Gamma}}(L^*, \Lambda^p).$$

Now we will prove analogical result for S^p.

Theorem 5.8: For every $a > 0$ and $j > 0$ we have

$$Tor_j^{\tilde{\Gamma}}(L^{*\otimes a}, S^p) \cong Tor_j^{\tilde{\Gamma}}(L^*, S^p).$$

Proof. First we will assume $a < p$. We will use Koszul and de Rham sequences as previously. We know that for $i \neq 0, p$:

$$L^{*\otimes a} \otimes_{\Gamma} (S^{p-i} \otimes \Lambda^i) \subset cr_p(L^{*\otimes a}) = 0$$

For $i = p$ we still have $L^{*\otimes a} \otimes_{\Gamma} \Lambda^p = 0$ but there is one case unknown: for $i = 0$ we have $L^{*\otimes a} \otimes_{\Gamma} S^p$. Let us have a look at the hyperhomology spectral sequences (two of them) with coefficients in the Koszul complex.

$${}^{I}E^2_{i,j} = Tor_i^{\tilde{\Gamma}}(L^{*\otimes a}, H_j(\mathbf{K^p})) \Rightarrow HTor_{i+j}(L^{*\otimes a}, \mathbf{K^p})$$

and

$${}^{II}E^2_{i,j} = H_i(Tor_j^{\tilde{\Gamma}}(L^{*\otimes a}, \mathbf{K^p})) \Rightarrow HTor_{i+j}(L^{*\otimes a}, \mathbf{K^p})$$

As previously, the first one converges to zero since Koszul complex is acyclic, so does the second. On the other hand second spectral sequence has possibly nontrivial terms in columns number 0 and p. The only non-trivial (possibly) differential on ${}^{II}E^p_{i,j}$ stage induces an isomorphism for $j \geq 0$:

$$d^p_{p,j} : Tor_j^{\tilde{\Gamma}}(L^{*\otimes a}, \Lambda^p) \cong Tor_{j+p-1}^{\tilde{\Gamma}}(L^{*\otimes a}, S^p) \qquad [5.7.1]$$

what implies that for $p - 2 \geq i \geq 0$ we have:

$$Tor_i^{\tilde{\Gamma}}(L^{*\otimes a}, S^p) = 0$$

Hence taking $i = 0$ we find the mysterious term $L^{*\otimes a} \otimes_{\Gamma} S^p$ trivial as well. Now corollary 1.6 and formula [5.7.1] for $a = 1$ gives us desired isomorphism:

$$Tor_{j+p-1}^{\tilde{\Gamma}}(L^*, S^p) \cong Tor_j^{\tilde{\Gamma}}(L^*, \Lambda^p) \cong Tor_j^{\tilde{\Gamma}}(L^{*\otimes a}, \Lambda^p) \cong Tor_{j+p-1}^{\tilde{\Gamma}}(L^{*\otimes a}, S^p)$$

Case $a = p$ is more complicated since we have to deal with non-trivial terms $L^{*\otimes p} \otimes_{\Gamma} (S^{p-i} \otimes \Lambda^i)$. This time we turn our attention to the hyperhomology spectral sequences with coefficients in the de Rham complex first. The first spectral sequence reduces to only two groups by the formula for the homology of the de Rham complex, so the spectral sequence collapses and we have:

$${}^{I}E^{\infty}_{p-1,0} = {}^{I}E^2_{p-1,0} = Tor_0^{\tilde{\Gamma}}(L^{*\otimes p}, L) = F_p, \quad \text{and} \quad {}^{I}E^{\infty}_{p,0} = {}^{I}E^2_{p,0} = Tor_0^{\tilde{\Gamma}}(L^{*\otimes p}, L) = F_p$$

The second spectral sequence ${}^{II}E^2_{i,j}$ has two non-trivial columns: $Tor_j^{\tilde{\Gamma}}(L^{*\otimes p}, S^p)$ stands in column $i = p$ and $Tor_j^{\tilde{\Gamma}}(L^{*\otimes p}, \Lambda^p)$ in ${}^{II}E^1_{0,j}$. Now we have also nontrivial row ${}^{II}E^1_{i,0}$. Actually this row is simply de Rham complex tensored over Γ with right Γ-module $L^{*\otimes p}$. On the next stage of spectral sequence we calculate homology of this complex. First we know that tensor product is right exact so ${}^{II}E^2_{0,0} = 0$. Then we know from corollary 1.6 and theorem 5.1 that non-trivial terms appear in column ${}^{II}E^2_{0,j}$ only for $(2s+1)(p-1)+1$ and $(2s+1)(p-1)$. This spectral sequence converges to $HTor_k$ which is nonzero for $k = p$

and $k = p-1$ only. That means differentials $d^k_{k,0}$ are isomorphisms for $1 < k < p-1$. In consequence ${}^{II}E^2_{k,0} = 0$ for $0 \leq k \leq p-2$. Now we need to examine carefully:

$$d^1_{p,0} : L^{*\otimes p} \otimes_\Gamma S^p \to L^{*\otimes p} \otimes_\Gamma (S^{p-1} \otimes L).$$

We claim it is monomorphism. Indeed, $L^{*\otimes p} \otimes_\Gamma S^p$ is one dimensional space. $cr_p(L^{*\otimes p})$ is just $F_p[\Sigma_p]$, so one can check directly that the following isomorphism holds:

$$cr(L^{*\otimes p}) \otimes_\Omega cr(S^p) \cong cr_p(L^{*\otimes p}) \otimes_{\Sigma_p} cr_p(S^p) \cong cr_p(S^p)$$

Similarly

$$cr_p(L^{*\otimes p}) \otimes_{\Sigma_p} cr_p(S^{p-1} \otimes L) \cong cr_p(S^{p-1} \otimes L).$$

Now

$$d^1_{p,0}(x_1 ... x_p) = \sum_{i=1}^{p} x_1 ... \hat{x_i} ... x_p \otimes x_i \in cr_p(S^{p-1} \otimes L)$$

and it is obviously a monomorphism. We have shown that ${}^{II}E^2_{p,0} = 0$. Now we examine carefully an exact sequence, induced by the differentials on ${}^{II}E^{p-1}_{i,j}$ and ${}^{II}E^p_{i,j}$ stages:

$$d^p_{p,j} : Tor^{\tilde{\Gamma}}_j(L^{*\otimes p}, S^p) \to Tor^{\tilde{\Gamma}}_{j+p-1}(L^{*\otimes p}, \Lambda^p)$$

For $j = 0$ we have:

$$0 \to {}^{II}E^\infty_{p,0} \to {}^{II}E^p_{p,0} = 0 \to Tor^{\tilde{\Gamma}}_{p-1}(L^{*\otimes p}, \Lambda^p) = {}^{II}E^p_{0,p-1} \to {}^{II}E^\infty_{0,p-1} \to 0.$$

In consequence F_p in $Tor^{\tilde{\Gamma}}_{p-1}(L^{*\otimes p}, \Lambda^p) = {}^{II}E^p_{0,p-1}$ survives to infinity and ${}^{II}E^2_{p-1,0} = 0$. This way we have shown that complex ${}^{II}E^1_{0,j}$ is acyclic. Moreover, F_p in $Tor^{\tilde{\Gamma}}_p(L^{*\otimes p}, \Lambda^p) = {}^{II}E^p_{0,p}$ also survives to infinity since ${}^{II}E^p_{p,0} = 0$. Now we look at exact sequence induced by $d^p_{p,j}$ for $j = 1$:

$$0 \to {}^{II}E^\infty_{p,1} = 0 \to {}^{II}E^p_{p,1} \to Tor^{\tilde{\Gamma}}_p(L^{*\otimes p}, \Lambda^p) = {}^{II}E^p_{0,p} \to {}^{II}E^\infty_{0,p} \to 0.$$

Last arrow is an isomorphism so $Tor^{\tilde{\Gamma}}_1(L^{*\otimes p}, S^p) = 0$. Differentials $d^p_{p,j}$ for $j > 1$ are isomorphisms so we get:

$$d^p_{p,j} : Tor^{\tilde{\Gamma}}_j(L^{*\otimes p}, S^p) \cong Tor^{\tilde{\Gamma}}_{j+p-1}(L^{*\otimes p}, \Lambda^p)$$

For $j > 1$ we have also the isomorphism from the proof of theorem 5.1:

$$Tor^{\tilde{\Gamma}}_j(L^*, S^p) \cong Tor^{\tilde{\Gamma}}_{j+p-1}(L^*, \Lambda^p),$$

so finally for $j > 1$ we get:

$$Tor^{\tilde{\Gamma}}_j(L^{*\otimes p}, S^p) \cong Tor^{\tilde{\Gamma}}_j(L^*, S^p)$$

This completes the proof.

There is one thing we want to underline here for further considerations. On the first stage of second spectral sequence with coefficients in Koszul complex, we have two non-trivial columns: $Tor_j^{\tilde{\Gamma}}(L^{*\otimes p}, S^p)$ stands in ${}^{II}E^1_{0,j}$ and $Tor_j^{\tilde{\Gamma}}(L^{*\otimes p}, \Lambda^p)$ in p^{th}column. Now we have also non-trivial row ${}^{II}E^1_{i,0}$. We know that on the second stage ${}^{II}E^1_{0,0} = 0$ since tensor product is right exact. We have already shown that ${}^{II}E^1_{0,j} = 0$ for $j < p-1$. Spectral sequence converges to zero so ${}^{II}E^1_{i,0} = 0$ for $i \leq p$. This discussion and the observations from the proof of 5.7 can be summarized in the statement that tensoring over Γ both Koszul and de Rham sequences with right Γ-module $L^{*\otimes p}$ leads to acyclic complexes of vector spaces.

Theorem 5.9: If $1 < a < p$ and $1 < i < a$ then:

$$Tor_j^{\tilde{\Gamma}}(S^{a-i} \otimes \Lambda^i, \Lambda^p) = 0$$

Proof. We know that for $a < p$, $S^{a-i} \otimes \Lambda^i$ is direct summand in $L^{*\otimes a}$. In the previous proof we have shown that tensoring any term of Koszul and de Rham complexes with $L^{*\otimes a}$ gives zero. Tensoring with direct summand must be trivial as well. The spectral sequences have exactly the same shape as in proof of theorem 5.1 for $a > 1$.

Corollary 5.10: If $1 < a < p$ and $1 < i < a$ then:

$$Tor_j^{\tilde{\Gamma}}(S^{a-i} \otimes \Lambda^i, S^p) = 0$$

Theorem 5.11: If $1 < a < p$ then:

$$Tor_j^{\tilde{\Gamma}}(S^a, \Lambda^p) = \begin{cases} F_p & \text{for } j = (2s+1)(p-1)+1, \; s \geq 0 \\ F_p & \text{for } j = (2s+1)(p-1), \; s \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$Tor_j^{\tilde{\Gamma}}(S^{a-1} \otimes \Lambda^1, \Lambda^p) = \begin{cases} F_p & \text{for } j = (2s+1)(p-1)+1, \; s \geq 0 \\ F_p & \text{for } j = (2s+1)(p-1), \; s \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. This time spectral sequences are exactly as in the proof of theorem 5.1 for $a = 1$ since $S^a \otimes_\Gamma L = F_p$ and $S^{a-1} \otimes \Lambda^1 \otimes_\Gamma L = F_p$ give nontrivial terms in the hyperhomology spectral sequence with coefficients in the de Rham complex.

Corollary 5.12: If $1 < a < p$ then:

$$Tor_j^{\tilde{\Gamma}}(S^a, S^p) = \begin{cases} F_p & \text{for } j = (2s+2)(p-1)+1, \; s \geq 0 \\ F_p & \text{for } j = (2s+2)(p-1), \; s \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$Tor_j^{\tilde{\Gamma}}(S^{a-1}\otimes\Lambda^1, S^p) = \begin{cases} F_p & \text{for } j=(2s+2)(p-1)+1,\ s\geq 0 \\ F_p & \text{for } j=(2s+2)(p-1),\ s\geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 5.13: If $1 < i < p$ then:

$$Tor_j^{\tilde{\Gamma}}(S^{p-i}\otimes\Lambda^i, \Lambda^p) = 0$$

Proof. We know that for $0 < i < p$, $S^{p-i}\otimes\Lambda^i$ is direct summand in $L^{*\otimes p}$. We have shown that tensoring over Γ both Koszul and de Rham sequences with right Γ-module $L^{*\otimes p}$ leads to acyclic complexes. Tensoring with some direct summand must give the same result. The spectral sequences have exactly the same shape as in proof of theorem 5.1 for $a > 1$.

Corollary 5.14: If $1 < i < p$ then:

$$Tor_j^{\tilde{\Gamma}}(S^{p-i}\otimes\Lambda^i, S^p) = 0$$

Theorem 5.15:

$$Tor_j^{\tilde{\Gamma}}(S^{p-1}\otimes\Lambda^1, \Lambda^p) = \begin{cases} F_p & \text{for } j=(2s+1)(p-1)+1,\ s\geq 0 \\ F_p & \text{for } j=(2s+1)(p-1),\ s\geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. This time spectral sequences are exactly as in Theorem 5.1 for $a = 1$ since two vector spaces $(S^{p-1}\otimes\Lambda^1)\otimes_\Gamma L = F_p$ give nontrivial terms in the hyperhomology spectral sequence with coefficients in the de Rham complex. Actually this theorem could be proved using theorem 1.5.

Corollary 5.16:

$$Tor_j^{\tilde{\Gamma}}(S^{p-1}\otimes\Lambda^1, S^p) = \begin{cases} F_p & \text{for } j=(2s+2)(p-1)+1,\ s\geq 0 \\ F_p & \text{for } j=(2s+2)(p-1),\ s\geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Remark 5.17: We did not say anything about divided powers. But of course one can easily perform the same calculations for D^i as were presented for S^i. The only change needed here is to use duals in the sense of Kuhn of the Koszul and de Rham sequences, or to combine at Λ^p Koszul complex and its dual and get an exact sequence connecting S^p and D^p. The details are left to the interested reader.

References:

[BP] S.Betley, T.Pirashvili. Stable K-theory as a derived functor. *J. Pure and Applied Algebra* 96 (1994) 245-258.

[BS] S.Betley, J.Slominska. New approach to the groups $H_*(\Sigma_n, Lie_n)$ by the homology theory of the category of functors. *J. Pure and Applied Algebra* 161 (2001) 31-43.

[B1] S.Betley. Stable derived functors, the Steenrod algebra and homological algebra in the category of functors. *Fundamenta Math.* 168 (2001) 279-293.

[B2] S.Betley. Ext groups for the composition of functors. Algebraic Topology: Categorical Decomposition Techniques. Birkhauser Verlag, Basel, Switzerland, *Progress in Math.* 215 (2003) 31-45.

[FLS] V.Franjou, J.Lannes, L.Schwartz. Autour de la cohomologie de MacLane des corps finis. *Invent. Math.* 115 (1994) 513-538.

[FFSS] V.Franjou, E.Friedlander, A.Scorichenko, A.Suslin. General linear and functor cohomology over finite fields. *Ann. of Math.* 150 (1999) 663-728.

[K] N.Kuhn Generic representation theory of the finite general linear groups and the Steenrod algebra: I. Amer. J. Math. 116 (1994) 327-360.

[P1] T.Pirashvili. Hodge decomposition for higher order Hochschild homology. *Ann. Scient. Ec. Norm. Sup.* 33 (2000) 151-179.

[P2] T.Pirashvili. Dold-Kan type theorem for Γ-groups. *Math. Annalen* 318 (2000) 277-298.

[Ri] B. Richter. Taylor towers for Γ-modules. *Annales de L'Institut Fourier* 51 (2001) 995-1023.

[Ro] A. Robinson. Gamma homology, Lie representations and E_∞ multiplications.*Inventiones Math. 152 (2003) 331-348.*

First author's address:
Institute of Mathematics of the Polish Academy of Sciences,
ul. Sniadeckich 8, 00-956 Warsaw 10, Poland
e-mail: jant@impan.gov.pl

Second author's address:
Instytut Matematyki, University of Warsaw
ul. Banacha 2, 02-097 Warsaw, Poland
e-mail: betley@mimuw.edu.pl