ON HOM AND EXT GROUPS BETWEEN PLETHYSMS

STANISŁAW BETLEY AND MARCIN CHAŁUPNIK

Introduction

In the present article we investigate Hom and Ext groups between compositions of functors (called in representation theory *plethysms*) in the category $\mathcal{P}$ of strict polynomial functors over a field of positive characteristic p. This topic was heavily studied in the case of composition with the Frobenius twist functor $I^{(1)}$ (see [FS], [FFSS], [T1], [C3], [C4]). This particular case is important, because precompositions of the form $F \circ I^{(1)}$ appear in the classical Franjou-Friedlander-Scorichenco-Suslin theorem [FFSS] comparing $\mathcal{P}$ with the category of $\mathbf{k}[GL_n(\mathbf{k})]$-mod, and at the time being is quite well understood. However the case of $I^{(1)}$ is very special, since we precompose with an additive functor, which greatly simplifies computations. On the other hand, the case of non-additive functor is largely unexplored (we can only point out for [Tr], where some computations of the groups $\mathrm{Ext}^*(I, F \circ G)$ in the category $\mathcal{F}$ of ordinary functors are obtained).

In our paper we start a systematic investigation of Ext and Hom groups of the form

$$\mathrm{Ext}^*_{\mathcal{P}}(G \circ F, H \circ F).$$

Our focus is homological algebra, hence we work over a field $\mathbf{k}$ of positive characteristic p. However, the reader should be aware of the complexity of the situation even in characteristic zero. Namely, for example the celebrated Foulkes conjecture [F] predicts the existence of embedding:

$$S^b \circ S^a \subset S^a \circ S^b$$

for $a > b$. Since we are now in characteristic zero, it would suffice to show that any irreducible character appearing in the LHS also appears in the RHS, but still the conjecture is open for $b > 3$. Hence it is unrealistic to expect simple closed formulas for, say, Hom-groups between compositions of symmetric powers. But still the possibility of applications to the problems like the Foulkes conjecture was one of the main motivation for the current work. Namely, we hope that, like in many instances in algebraic geometry, the positive characteristic case can help understanding the situation in characteristic zero. More concretely, as we will see in our article, the technology utilizing the de Rham complex and the Cartier theorem provides link between Hom and Ext groups for larger and smaller symmetric powers allowing inductive arguments of all sorts. Also some rich combinatorial/simplicial structures we encountered in Section 1 seem to us to be quite universal and characteristic free. We find them very promising as a subject of a further research. Having said this, we do not claim any applications of the results of our paper to the classical problems in characteristic zero. We rather think of our article as the one which besides establishing some non-trivial yet quite particular results, sets the framework for further research, identifies the structures governing computations, tests the limits of known methods (Koszul and de Rham complexes...) and merges them with the ideas specific to our situation like modular Hecke algebras.

Now, let us discuss the contents of our article. The results and methods used in the paper are quite diverse and can be sorted out in two ways. The first part of the paper (Sections 1 and 2) concerns what we call "additive structures", ie. we compute ranks of the Hom and Ext groups between plethysms in $\mathcal{P}$. In the second part (Sections 3 and 4) we investigate extra structures existing on Hom/Ext groups: we determine the Yoneda multiplication for some plethysms in Section 3 and we describe the behavior of the adjunction generated by the operation of precomposing with S^2. However, inside each part there is a stark contrast between the methods and results concerning Hom groups, which are largely combinatorial and characteristic free and those concerning higher Exts, where a heavy use of spectral sequences and the de Rham/Koszul complexes is mandatory. This is the reason why we relegated the Hom computations to separate Section 1, and also we divided Sections 3 and 4 into subsections. We also point out that in the course of paper we impose more and more restrictions and specializations on our results. While results of Section 1.1 hold for a large class of plethysms and are valid over any field, starting from Section 1.2 we assume the ground field $\mathbf{k}$ has a positive characteristic p and we specialize to the plethysms of the form $S^p \circ F$ for $F \in \mathcal{P}_i$ for $i < p$. Then the multiplicative results of Section 3 concern plethysms $S^p \circ S^2$ and also the main result of Section 4 describes the unit of adjunction generated by the (derived) functor

$$\mathbf{C}_{S^2} : \mathcal{D}P_p \longrightarrow \mathcal{D}P_{2p}$$

of precomposing with S^2. Thus the strongest results of our article concern the plethysms with S^2, which is the simplest possible nonadditive functor. This situation, as we will see is already quite rich and interesting which shows again how complex and challenging may be the understanding plethysms with general nonadditive functors.

Now we shall guide the reader through the main results of the paper. As we have mentioned we study Hom-groups in Section 1. Many important objects of $\mathcal{P}_d$ are obtained from the tensor power functor I^d by applying an operation (functor) using the action of the symmetric group Σ_d on it. Thus we start by investigating an interplay between strict polynomial functors and finite group action in Section 1.1. We show in Theorem 1.1 when one can " go with a group action inside the Hom-groups", which gives some general formula (Corollary 1.2) for Hom-groups in $\mathcal{P}$. We would like turn the reader attention to the fact that Corollary 1.2 is the source and precursor of the connection between Hom-groups in $\mathcal{P}$ and the Hecke algebras which will be vital in the further parts of article. In Section 1.2 we specialize to the main subject of the paper ie. the compositions of functors of the form $S^d \circ F$, where $F \in \mathcal{P}_i$ for $i < p$. We again start with some general (Corollary 1.7), but it becomes really interesting for $F = S^i$, where it can be interpreted in terms of graphs and simplicial sets. The main result is Theorem 1.10 which interprets (the $\mathbf{k}$-basis of) $\mathrm{Hom}_{\mathcal{P}}(S^d \circ S^2, S^d \circ S^2)$ in terms of bipartite graphs and thus computes its rank. The rest of Section 1 is devoted to the study of the groups $\mathrm{Hom}_{\mathcal{P}}(S^d \circ S^i, S^d \circ S^i)$ for $i > 2$. In Proposition 1.12 we refine its combinatorial description by introducing certain combinatorial object $Polyh(d, i)$ consisting of multisets, which can be nicely interpreted as some simplicial sets. We find this interpretation interesting and promising, yet the classification problem which should be solved in order to obtain the generalization of Theorem 1.10 for $i > 2$ seems to be very challenging in general. We hope to tackle it in a future work.

In Section 2 we fix $\mathbf{k}$ to be a field of characteristic p and study the entire graded group $\mathrm{Ext}^*_{\mathcal{P}}(S^p \circ F, S^p \circ F)$ for $F \in \mathcal{P}_i$ with $i < p$. The main result, Theorem 2.9 describes this group in terms of the graded space $F^* = \mathrm{Ext}^*_{\mathcal{P}_{pi}}(F^{(1)}, F^{(1)})$ (we point out that the graded group F^* can be effectively computed thanks to the Collapsing Conjecture [C3]).

In Section 3 we describe the Yoneda multiplication in $\mathrm{Ext}^*_{\mathcal{P}}(S^p \circ S^2, S^p \circ S^2)$. As we have already mentioned, the descriptions of Hom and $\mathrm{Ext}^{>0}$ are quite different. We describe the multiplication in higher Exts in Theorem 3.1, which requires a careful study of the relevant spectral sequences, while we interpret $\mathrm{Hom}_{\mathcal{P}}(S^p \circ S^2, S^p \circ S^2)$ as a certain modular Hecke algebra in Proposition 3.7 (which is again a characteristic free result). Then, which is the most challenging part of Section 3, we describe in Theorem 3.8 the multiplication between Hom and $\mathrm{Ext}^{>0}$. The answer is given in terms of the representation of Hecke algebra called index (the necessary background on Hecke algebra is provided in Section 4.2).

Then in Section 4 we analyze the unit $\mathbf{RU}$ of the adjunction:

$$\mathbf{C}_{S^2} : \mathcal{D}P_p \longrightarrow \mathcal{D}P_{2p},$$

$$\mathbf{K}_{S^2} : \mathcal{D}P_{2p} \longrightarrow \mathcal{D}P_p,$$

($\mathbf{C}_{S^2}$ possesses the right adjoint by the Special Adjoint Functor theorem, and adjunction carries over to the derived categories). It was shown in [C3] that the analogous unit for the precomposition with $I^{(1)}$ has an explicit and quite simple description. It had strong computational applications (the so called Collapsing conjecture) and also allowed in [C4] to produce some interesting subcategories of $\mathcal{D}P_d$. Now it appears that this phenomenon is not bound to the case of $I^{(1)}$. Namely we explicitly describe in Theorem 4.10 the unit of the adjunction in $\mathcal{D}P_p$ for the precomposition with S^2 (we also provide the description of the unit in the semisimple case in all degrees (Theorem 4.3), which may be interesting for its own). This result, apart from possible computational applications, also sheds some light onto the dichotomy between Homs and higher Exts empirically observed in the previous sections. Namely, we see that **RU** decomposes into a large characteristic free part and a small derived component, for which the extremely simple p-local structure of the symmetric group Σ_p accounts. This explains why in the computations we usually have much larger Hom-groups than higher Ext-groups.

1. Hom between plethysms

In this section we describe the additive structure of Hom-groups for a large class of functors, which includes some plethysms. In Section 1.1 we study an interplay between $\mathcal{P}_d$ and group actions and produce certain general formula for Hom-groups. In Section 1.2 we specialize to the case of plethysms, where some nice combinatorial descriptions can be obtained.

1.1. **Hom-groups and group actions.** For a finite group Σ we say that $F \in \mathcal{P}_d$ is Σ-equivariant when for any V, $F(V)$ is equipped with a Σ-action and for any map $f : V \longrightarrow W$ the induced map $F(f) : F(V) \longrightarrow F(W)$ is Σ-equivariant. Thus when α is a $\mathbf{k}$-linear functor from the category of finite dimensional $\mathbf{k}[\Sigma]$-modules to the category of finite dimensional $\mathbf{k}$-modules, the assignment $V \mapsto \alpha(F(V))$ defines a new functor $\alpha(F) \in \mathcal{P}_d$. Also, for any $G \in \mathcal{P}_d$, α acts on $\mathrm{Hom}_{\mathcal{P}_d}(G, F)$. Our goal is to compare $\mathrm{Hom}_{\mathcal{P}_d}(G, \alpha(F))$ and $\alpha(\mathrm{Hom}_{\mathcal{P}_d}(G, F))$.

Theorem 1.1. *Let F, G, α be as above. Then there exists a natural map $\mathrm{Hom}_{\mathcal{P}_d}(G, \alpha(F)) \longrightarrow \alpha(\mathrm{Hom}_{\mathcal{P}_d}(G, F))$ which is an isomorphism if one of the assumptions is satisfied:*

- *G is projective.*
- *α is left exact.*

Dually, there exists a natural map $\alpha^{\#}(\mathrm{Hom}_{\mathcal{P}_d}(F, G)) \longrightarrow \mathrm{Hom}_{\mathcal{P}_d}(\alpha(F), G)$, where $\alpha^{\#}$ is the Kuhn dual of α, ie. $\alpha^{\#}(M) := (\alpha(M^*))^*$. The above map is an isomorphism if one of the following assumptions is satisfied:

- *G is injective.*
- *α is right exact.*

Proof: Assume first that $G = \Gamma^{d,U}$ (we will use frequently in our article functors with parameters (see eg. [C3, Section 2]) following the conventions: $F^U(V) := F(U^* \otimes V)$ and $F_U(V) := F(U \otimes V)$). Then by the Yoneda lemma we have $\alpha(\mathrm{Hom}_{\mathcal{P}_d}(G, F)) = \alpha(F(U) = \alpha(F)(U) = \mathrm{Hom}_{\mathcal{P}_d}(G, \alpha(F))$. Now, let G be any projective object. Then we have a split exact sequence:

$$\Gamma^{d,U'} \longrightarrow \Gamma^{d,U} \longrightarrow G \longrightarrow 0.$$

Then by applying $\mathrm{Hom}_{\mathcal{P}_d}(-, \alpha(F))$ we obtain the exact sequence:

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{P}_d}(G, \alpha(F)) \longrightarrow \mathrm{Hom}_{\mathcal{P}_d}(\Gamma^{d,U}, \alpha(F)) \longrightarrow \mathrm{Hom}_{\mathcal{P}_d}(\Gamma^{d,U'}, \alpha(F)).$$

Similarly, by applying $\mathrm{Hom}_{\mathcal{P}_d}(-, F)$ we obtain the exact sequence:

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{P}_d}(G, F) \longrightarrow \mathrm{Hom}_{\mathcal{P}_d}(\Gamma^{d,U}, F) \longrightarrow \mathrm{Hom}_{\mathcal{P}_d}(\Gamma^{d,U'}, F).$$

Since the resulting sequence is split exact, it remains exact after applying α, hence we get the exact sequence:

$$0 \longrightarrow \alpha(\mathrm{Hom}_{\mathcal{P}_d}(G,F)) \longrightarrow \alpha(\mathrm{Hom}_{\mathcal{P}_d}(\Gamma^{d,U},F)) \longrightarrow \alpha(\mathrm{Hom}_{\mathcal{P}_d}(\Gamma^{d,U'},F)).$$

Thus, using the functoriality of isomorphisms established for representable G we get a commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \alpha(\mathrm{Hom}_{\mathcal{P}_d}(G,F)) & \longrightarrow & \alpha(\mathrm{Hom}_{\mathcal{P}_d}(\Gamma^{d,U},F)) & \longrightarrow & \alpha(\mathrm{Hom}_{\mathcal{P}_d}(\Gamma^{d,U'},F)) \\ & & & & \wr\mid & & \wr\mid \\ 0 & \longrightarrow & \mathrm{Hom}_{\mathcal{P}_d}(G,\alpha(F)) & \longrightarrow & \mathrm{Hom}_{\mathcal{P}_d}(\Gamma^{d,U},\alpha(F)) & \longrightarrow & \mathrm{Hom}_{\mathcal{P}_d}(\Gamma^{d,U'},\alpha(F)). \end{array}$$

Hence, by diagram chasing we obtain an isomorphism $\alpha(\mathrm{Hom}_{\mathcal{P}_d}(G,F)) \simeq \mathrm{Hom}_{\mathcal{P}_d}(G,\alpha(F))$.
Now, consider an arbitrary $G \in \mathcal{P}_d$. Then by repeating the above constructions we arrive at the commutative diagram with the bottom row exact:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \alpha(\mathrm{Hom}_{\mathcal{P}_d}(G,F)) & \longrightarrow & \alpha(\mathrm{Hom}_{\mathcal{P}_d}(\Gamma^{d,U},F)) & \longrightarrow & \alpha(\mathrm{Hom}_{\mathcal{P}_d}(\Gamma^{d,U'},F)) \\ & & & & \wr\mid & & \wr\mid \\ 0 & \longrightarrow & \mathrm{Hom}_{\mathcal{P}_d}(G,\alpha(F)) & \longrightarrow & \mathrm{Hom}_{\mathcal{P}_d}(\Gamma^{d,U},\alpha(F)) & \longrightarrow & \mathrm{Hom}_{\mathcal{P}_d}(\Gamma^{d,U'},\alpha(F)). \end{array}$$

This time by a diagram chasing we only obtain a map: $\alpha(\mathrm{Hom}_{\mathcal{P}_d}(G,F)) \longrightarrow \mathrm{Hom}_{\mathcal{P}_d}(G,\alpha(F))$. However, when α is left exact, also the top row is exact, hence the above map is an isomorphism.
The proof of the dual statement is analogous. □

Let us now discuss some examples. The source of many examples is a certain observation used empirically in [C1] and conceptualized and related to the Auslander theory in [FP]. Namely, one can see that many important objects in $\mathcal{P}_d$ can be obtained from the Σ_d-invariant object I^d by applying certain functors from the category of finite dimensional $\mathbf{k}[\Sigma_d]$-modules to the category of finite dimensional $\mathbf{k}$-modules. The most obvious examples are: the symmetric power functor S^d is obtained from I^d by applying the functor of Σ_d-coinvariants, or the divided power functor Γ^d is obtained from I^d by applying the functor of Σ_d-invariants. Less trivially, for a Young diagram λ, the Schur functor S_λ can be obtained from I^d by applying certain functor s_λ (see [C1]). On the other hand, it was shown in [C1] that the Specht module Sp_λ is $s_\lambda(\mathbf{k}[\Sigma_d]$. Since I^d is projective, by applying Theorem 1.1 to $F = G = I^d$ and $\alpha = s_\lambda$ and obtain:

$$\mathrm{Hom}_{\mathcal{P}_d}(I^d, S_\lambda) \simeq \mathrm{Hom}_{\mathcal{P}_d}(I^d, s_\lambda(I^d)) \simeq s_\lambda(\mathrm{Hom}_{\mathcal{P}_d}(I^d,I^d)) \simeq s_\lambda(\mathbf{k}[\Sigma_d]) \simeq Sp_\lambda$$

hence we obtain the well known description of the Specht modules in terms of "the Schur functor" (see [M, Chapter 4]).

Let H, K be subgroups of Σ_d. We consider the right exact functors $(-)_H$, $(-)_K$ of coinvariants with respect to H and K, and their Kuhn duals which are the functors $(-)^H$, $(-)^K$ of invariants. By applying the both parts of Theorem 1.1 we get:

Corollary 1.2. *For any $H, K \leq \Sigma_d$ we have:*

$$\mathrm{Hom}_{\mathcal{P}_d}((I^d)_H, (I^d)_K) \simeq {}^H(\mathbf{k}[\Sigma_d]_K).$$

Let us formalize the construction appearing in Cor. 1.2.

Definition 1.3. *Let M be a H-K-bimodule. We call the $\mathbf{k}$-module ${}^H(M_K)$ the Hecke space associated to M.*

We will discuss multiplicative properties of the above construction and explain terminology in Section 3.2. Now we only provide its $\mathbf{k}$-basis.

Proposition 1.4. *Assume that there exists X, a $\mathbf{k}$-basis of M preserved by the both actions of H and K. Then there is an isomorphism of vector spaces:*

$$^H(M_K) \simeq \mathbf{k}[H\backslash X/K],$$

where $H \setminus X/K$ is the set of $H-K$ double cosets on X ie. we factorize X through the relation: $x \sim y$ when there exist $h \in H$, $k \in K$ such that $y = h \cdot x \cdot k$.

Proof: Since M as $H-K$-bimodule is isomorphic to the direct sum of the sub-bimodules generated by the elements of X belonging to the same double coset, it suffices to show that when X has a single double coset, the Hecke space is one-dimensional. So, assume that $H \setminus X/K$ is one-element and let $\{x_1, \ldots, x_n\}$ be the set of representatives for the right K-action on X. Then it may be thought of as a basis for M_K, and since they all belong to the same double coset, we see that M_K is a transitive permutative left H-module. Hence its space of invariants is one-dimensional □

1.2. **The case of plethysms.** In this subsection we investigate plethysms of the form $S^d \circ F$, for $F \in \mathcal{P}_i$ for $i < p$. First we recall the well known fact:

Proposition 1.5. *Any $F \in \mathcal{P}_i$ for $i < p$ is of the form $f(I^d)$ for some exact functor $f : \mathbf{k}[\Sigma_d]-\text{mod} \longrightarrow \mathbf{k}-\text{mod}$.*

Proof: This fact follows from the machinery of [FP], hence essentially from the Auslander theory, but we can also give a simple elementary argument. Namely, for $i < p$, $\mathcal{P}_i$ is semisimple (see eg. [G], in fact the proof in characteristic zero going back to Schur still works here). Thus it suffices to provide f for a simple functor F. But the simple functors in $\mathcal{P}_i$ are just Schur functors S_λ for which we have s_λ. But s_λ is a composite of invariant and coinvariant functors in the category of $\mathbf{k}[\Sigma_i]$ modules, which are clearly exact. □

Now we would like to describe a plethysm $S^d \circ F$ as $\alpha_F(I^{di})$. For this we need some elementary facts on prolonging functors.

Lemma 1.6. *Let $f : \mathbf{k}[\Sigma]-\text{mod} \longrightarrow \mathbf{k}-\text{mod}$ be a functor. Then:*

(1) *For any finite group Σ' the composite functor*

$$\mathbf{k}[\Sigma \times \Sigma']-\text{mod} \xrightarrow{res} \mathbf{k}[\Sigma]-\text{mod} \xrightarrow{f} \mathbf{k}-\text{mod}$$

naturally lifts to the functor $\mathbf{k}[\Sigma \times \Sigma']-\text{mod} \longrightarrow \mathbf{k}[\Sigma']-\text{mod}$, which will be denoted by $f \times \text{id}$.

(2) *f extends to the functor*

$$f^{\times d} : \mathbf{k}[\Sigma^{\times d}]-\mathrm{mod} \longrightarrow \mathbf{k}-\mathrm{mod}$$

such that $f^{\times d}$ restricted to any copy of $\mathbf{k}[\Sigma]-\mathrm{mod}$ is f.
Moreover, let for a $\mathbf{k}[\Sigma^{\times d}]$-module M and a permutation $\sigma \in \Sigma_d$, M^σ stands for the $\mathbf{k}[\Sigma^{\times d}]$-module with the $\Sigma^{\times d}$-action twisted by σ. Then there is a natural isomorphism $f^{\times d}(M) \simeq f^{\times d}(M^\sigma)$.

(3) *Let $\Sigma \wr \Sigma_d$ be the wreath product. Then f extends to the functor*

$$f \wr \mathrm{id} : \mathbf{k}[\Sigma \wr \Sigma_d]-\mathrm{mod} \longrightarrow \mathbf{k}[\Sigma_d]-\mathrm{mod}$$

such that $f \wr \mathrm{id}$ restricted to $\mathbf{k}[\Sigma^{\times d}]-\mathrm{mod}$ is $f^{\times d}$.

Proof: The first assertion follows from the fact that Σ' acts on a $\mathbf{k}[\Sigma \times \Sigma']$-module M via Σ-endomorphisms, hence f acts on them, thus yielding the Σ-action on $f(M)$.
For the second assertion we start with picking the first copy of Σ from $\Sigma^{\times d}$ and applying part (1) to the product $\Sigma \times \Sigma^{\times(d-1)}$, thus obtaining the functor

$$f \times \mathrm{id} : \mathbf{k}[\Sigma \times (\Sigma^{\times(d-1)})]-\mathrm{mod} \longrightarrow \mathbf{k}[\Sigma^{\times(d-1)}]-\mathrm{mod}.$$

Then we pick another copy of Σ from $\Sigma^{\times(d-1)}$ and repeat the procedure. Finally we arrive at the desired functor $f^{\times d}$. We also observe that the resulting functor does not depend (up to isomorphism) on the order of picking the factors from the product $\Sigma^{\times d}$. This also shows the invariance of $f^{\times}$ with respect to the action of the symmetric group Σ_d.
For the third part we first construct $f^{\times d}$ by using (2). Then we observe that for a $\mathbf{k}[\Sigma \wr \Sigma_d]$-module M, the multiplication by a permutation $\sigma \in \Sigma_d$ yields the $\Sigma^{\times d}$-homomorphism:

$$L_\sigma : M \longrightarrow M^\sigma.$$

Hence by applying $f^{\times d}$ and using the second part of (2) we obtain the map:

$$f^{\times d}(M) \stackrel{f^{\times d}(L_\sigma)}{\longrightarrow} f^{\times d}(M^\sigma) \simeq f^{\times d}(M),$$

which equips $f^{\times d}(M)$ with the Σ_d-action. □

Therefore for F of the form $f(I^i)$, the plethysm $S^d \circ F$ can be described as $\alpha_F(I^{di})$ for the functor $\alpha_F : \mathbf{k}[\Sigma_{di}]-\mathrm{mod} \longrightarrow \mathbf{k}-\mathrm{mod}$ given as the composite:

$$\mathbf{k}[\Sigma_{di}]-\mathrm{mod} \stackrel{res}{\longrightarrow} \mathbf{k}[\Sigma_i \wr \Sigma_d]-\mathrm{mod} \stackrel{f \wr \mathrm{id}}{\longrightarrow} \mathbf{k}[\Sigma_d]-\mathrm{mod} \stackrel{\otimes_{\Sigma_d}\mathbf{k}}{\longrightarrow} \mathbf{k}-\mathrm{mod}.$$

Since α_F is left exact, we get:

Corollary 1.7. *For any $F \in \mathcal{P}_i$ for $i < p$, we have:*

$$\mathrm{Hom}_{\mathcal{P}_{di}}(S^d \circ F, S^d \circ F) = \alpha^{\#}(\alpha(I^d)).$$

Now, let us look more closely at the special case of $F = S^i$. Then we can apply Cor. 1.6, but in order to obtain a more explicit description we shall divide the process into two steps, ie. we firstly factorize I^{di} through the action of $\Sigma_i^{\times d}$ to obtain $(S^i)^{\otimes d}$, and then we factorize $(S^i)^{\otimes d}$ through the action of Σ_d. Let $Gr_{bip}(d, i; d, i)$ stand for the set of 2-regular bipartite graphs with the set of vertices being the disjoint union of two copies of $[d] = \{1, \ldots, d\}$. This means that we impose on our graphs the following conditions: there are exactly i edges attached to any vertex (multiple edges allowed) and there are no edges between vertices belonging to the same copy of $[d]$. $Gr_{bip}(d,i;d,i)$ is equipped with the $\Sigma_d \times \Sigma_d$ action coming from renumbering vertices inside the copies of $[d]$. Then we have:

Proposition 1.8. *There are isomorphisms of* $\mathbf{k}[\Sigma_d]$*-bimodules:*

$$\mathrm{Hom}_{\mathcal{P}_{di}}((S^i)^{\otimes d}, S^i)^{\otimes d}) \simeq \mathbf{k}[\Sigma_i^{\times d} \backslash \Sigma_{di} / \Sigma_i^{\times d}] \simeq \mathbf{k}[Gr_{bip}(d,i;d,i)].$$

Proof: By the Exponential Formula (see [FFSS]), we know that $\mathrm{Hom}_{\mathcal{P}_{di}}((S^i)^{\otimes d}, (S^i)^{\otimes d})$ has a basis consisting of $d \times d$ matrices with entries from the set $\{0, 1, \ldots, i\}$ such that all the row and column sums are equal to i. Then for a matrix $A = [a_{st}]$ we form our graph by putting a_{st} edges between vertices s and t. The action of Σ_d on, respectively the source and target of endomorphisms corresponds to permuting respectively columns and rows of the matrix, hence after taking graphs we obtain the described above action on $Gr_{bip}(d,i;d,i)$. □

Then we can apply again Theorem 1.1 for $F = G = (S^i)^{\otimes d})$, and Prop. 1.4, 1.8 in order to describe $\mathrm{Hom}_{\mathcal{P}_{di}}(S^d \circ S^i, S^d \circ S^i)$ in terms of $\mathrm{Hom}_{\mathcal{P}_{di}}((S^i)^{\otimes d}, (S^i)^{\otimes d})$. Thus we clearly get:

Corollary 1.9. *We have an isomorphism:*

$$\mathrm{Hom}_{\mathcal{P}_{di}}(S^d \circ S^i, S^d \circ S^i) \simeq {}^{\Sigma_d}(\mathrm{Hom}_{\mathcal{P}_{di}}((S^i)^{\otimes d}, (S^i)^{\otimes d})_{\Sigma_d}) \simeq$$
$${}^{\Sigma_d}(\mathbf{k}[Gr_{bip}(d,i;d,i)]_{\Sigma_d}) \simeq \mathbf{k}[\Sigma_d \setminus Gr_{bip}(d,i;d,i)/\Sigma_d].$$

Therefore our task is to describe the set of Σ_d-double cosets on $Gr_{bip}(d,i;d,i)$, or equivalently the set of isomorphism classes of such graphs. We start with the case $i = 2$.

Theorem 1.10. *Two graphs in* $Gr_{bip}(d,2;d,2)$ *are isomorphic if and only if they have the same lengths of connected components.*
Therefore the dimension of $\mathrm{Hom}_{\mathcal{P}_{2d}}(S^d \circ S^2, S^d \circ S^2)$ *is equal to* $|\Lambda(d)|$*, the number of Young diagrams of weight* d*.*

Proof: This is an elementary exercise in combinatorics, but we shall divide the proof into two parts, since a similar approach sheds light onto the case $i > 2$. Namely, first we analyze the right action:

Lemma 1.11. *There is an isomorphism of* Σ_d*-sets:*

$$Gr_{bip}(d,2;d,2)/\Sigma_d \simeq Gr(d,2),$$

where $Gr(d,2)$ *stands for the set of graphs with the set of vertices* $[d]$*, with each vertex of degree* 2*.*

Proof of the Lemma: We construct the map $\Psi : Gr_{bip}(d,2;d,2) \longrightarrow Gr(d,2)$ by the following rule: we restrict ourselves to a one copy of $[d]$ and we connect two vertices by the edge if and only if they are connected by the path of length 2 in the original graph. Then it is obvious that Ψ is Σ_d-invariant, onto and $\Psi(X) = \Psi(Y)$ if and only if X and Y belong to the same orbit of the right Σ_d-action on $Gr_{bip}(d, 2; d, 2)$. □

Lemma reduces our task to the analysis of the isomorphism classes of graphs in $Gr(d, 2)$. However, any such a graph is a disjoint sum of cycles and its isomorphism class is determined by the lengths of its cycles. □

Let us now discuss the general case. Let $Polyh'(d, i)$ denote the set of d subsets with multiplicities of $[d]$ each consisting of i elements with multiplicities. When we additionally impose the condition that each number (counted with multiplicities) occurs i times, we denote the resulting set by $Polyh(d, i)$. Thus, eg. $Polyh(2, 3) = \{\{\{1, 1, 1\}, \{2, 2, 2\}\}, \{\{1, 1, 2\}, \{1, 2, 2\}\}\}$. We equip $Polyh(d, i)$ with the Σ_d-action by permuting numbers and we have:

Proposition 1.12. *There is an isomorphism of Σ_d-sets:*

$$(Gr_{bip}(d, i; d, i))_{\Sigma_d} \simeq Polyh(d, i).$$

Proof: This is an analog of Lemma 1.11, but the underlying combinatorics is much more complicated. For this reason, instead of giving a purely combinatorial proof, we will provide the one relying on Theorem 1.1 and the Exponential Formula. Namely, on the one hand we have:

$$\mathbf{k}[(Gr_{bip}(d, i; d, i))/\Sigma_d] \simeq (\mathrm{Hom}_{\mathcal{P}_{di}}((S^i)^{\otimes d}, (S^i)^{\otimes d})_{\Sigma_d} \simeq \mathrm{Hom}_{\mathcal{P}_{di}}((S^i)^{\otimes d}, S^d \circ S^i).$$

On the other hand one can compute the last group directly. Since $i < p$, $(S^i)^{\otimes d} \simeq, (\Gamma^i)^{\otimes d}$, while by the Yoneda lemma, for any $G \in \mathcal{P}_{di}$ we have

$$\mathrm{Hom}_{\mathcal{P}_{di}}((\Gamma^i)^{\otimes d}, G) \simeq G^{i,\ldots,i}(V_1 \oplus \ldots \oplus V_d),$$

for one-dimensional spaces V_i, and the superscript means the weight space for the weight $(i, \ldots, i)$, ie. the subfunctor of the functor in d variables:

$$(V_1, \ldots, V_d) \mapsto G(V_1 \oplus \ldots \oplus V_d)$$

of degree i with respect to all V_l's. Now let us describe $S^d \circ S^i(V_1 \oplus \ldots \oplus V_d)$. For this we need some notation. For $\gamma \in Polyh'(d, i)$ we write $\gamma = \{\gamma_1^{a_1}, \ldots, \gamma_k^{a_k}\}$, where γ_j is a subset (with multiplicities) of $[d]$ and a_j is its multiplicity (ie. $\sum_j a_j = d$). Then for each individual γ_j we put $\gamma_j(l)$ to be the multiplicity of l in γ_j. Then we have:

$$S^d \circ S^i(V_1 \oplus \ldots V_d)) \simeq$$

$$\bigoplus_{\gamma \in Polyh'(d,i)} S^{a_1}(S^{\gamma_1(1)}(V_1) \otimes \ldots \otimes S^{\gamma_1(d)}(V_d)) \otimes \ldots \otimes S^{a_k}(S^{\gamma_k(1)}(V_1) \otimes \ldots \otimes S^{\gamma_k(d)}(V_d)).$$

Now we see that for V_l's one-dimensional each summand is one-dimensional and that its degree with respect to V_l equals the number of occurrences of l in γ. This shows that the dimension of $\mathrm{Hom}_{\mathcal{P}_{di}}((S^i)^{\otimes d}, S^d \circ S^i)$ equals the cardinality of $Polyh(d, i)$. Moreover, since the Σ_d-action on Hom corresponds via the Yoneda lemma to renumbering V_l's, this bijection is Σ_d-equivariant. □

Thus in order to compute the dimension of $\mathrm{Hom}_{\mathcal{P}_{di}}((S^d \circ S^i, S^d \circ S^i)$ we need to count the orbits of the Σ_d-action on $Polyh(d, i)$. To feel the complexity of the problem we shall interpret $Polyh(d,i)$ geometrically. For $i = 2$, it is natural to associate to $\gamma \in Polyh'(d,2)$ a graph with vertices from $[d]$, namely we connect two numbers by an edge when they occur in some γ_j. Then it is easy to see that this construction gives a Σ_d-invariant bijection:

$$Polyh(d,2) \simeq Gr(d,2) \tag{1.1}$$

hence we recover Lemma 1.11. Then, analogously, for $i > 2$ one can associate to $\gamma \in Polyh'(d,i)$ a simplicial set generated by 0-simplices from $[d]$ and d-simplices γ_j. We emphasize the use of simplicial sets and not just polyhedra here, since if in γ_j there are higher multiplicities, then the simplex is degenerated. The condition defining $Polyh(d,i)$ geometrically means that any vertex belongs to exactly d d-dimensional simplices. Then our problem is related to that of classifying of isomorphism classes of such simplicial sets, which in general seems to be difficult. In fact if we would like to reflect the entire combinatorial structure in this framework, then we should also take into account the external multiplicities a_j. Thus the fully adequate language would be that of simplicial "sets with multiplicities" or bisimplicial sets with external structure degenerated, which would make the classification problem even harder.

2. Ext between plethysms

In this section we describe the additive structure of groups $\mathrm{Ext}_{\mathcal{P}_{pi}}(S^p \circ F, S^p \circ F)$, where F is a functor of degree i and $i < p$. As main tools for our calculations we use Koszul and de Rham sequences. This technique forces us to get Ext - calculations between other plethysms also. In the whole section our ground field is any field $\mathbf{k}$ of characteristic p.

2.1. **Kuhn's theorem revisited.** Let E, F and G be functors in $\mathcal{P}$. Precomposition with E yields an exact functor $\mathcal{P} \longrightarrow \mathcal{P}$. The goal of this section is to show that this functor induces a monomorphism on Ext-groups in $\mathcal{P}$. The analogous result was proved in the category $\mathcal{F}$ by Kuhn in [K]. We show that his approach works also in $\mathcal{P}$. Our argument is based on Touze's classification of additive functors in $\mathcal{P}$ obtained in [T1] and "twist injectivity" of Ext, see [FFSS, Corollary 1.3].

Theorem 2.1. *Let E, F and G be as above. Assume that E is a nonconstant functor. The precomposition with E induces an injective map*

$$\mathrm{Ext}^*_{\mathcal{P}}(F,G) \longrightarrow \mathrm{Ext}^*_{\mathcal{P}}(F \circ E, G \circ E).$$

Moreover this map splits naturally in F and G.

Proof: We will follow closely [K, proof of Theorem 4.8]. Let I denote the identity functor and let $I \oplus k$ denote the strict polynomial functor, which takes $V \in \mathcal{V}$ to $V \oplus k$, where k is our ground field and $\mathcal{V}$ denotes here the category of finite dimensional vector spaces over k. For a given $E \in \mathcal{P}$ let ΔE denote the cokernel of the split injection $E \to E \circ (I \oplus k)$. We can iterate this construction. As $E \in \mathcal{P}$ we know that E has finite Eilenberg-Mac Lane degree. If this degree is equal to d then $\Delta^{d-1}E$ is a functor of Eilenberg-Mac Lane degree 1. By [T, Section 3] we know that $\Delta^{d-1}E = k^a \oplus H$ for certain natural a and H satisfying

$$0 \neq H = \bigoplus_{j\in J} I^{(n_j)}$$

where for any natural n, $I^{(n)}$ is the identity functor twisted n times by the Frobenius homomorphism. Obviously $\Delta^{d-1}E$ is a direct summand in $E\circ(I\oplus k^{d-1})$. Then for a certain natural n we have the sequence of homomorphisms of Ext-groups:

$$\mathrm{Ext}^*_{\mathcal{P}}(F,G) \longrightarrow \mathrm{Ext}^*_{\mathcal{P}}(F\circ E, G\circ E) \longrightarrow \mathrm{Ext}^*_{\mathcal{P}}(F\circ E\circ(I\oplus k^{d-1}), G\circ E\circ(I\oplus k^{d-1}))$$
$$... \longrightarrow \mathrm{Ext}^*_{\mathcal{P}}(F\circ I^{(n)}, G\circ I^{(n)})$$

where the last map is induced by a projection and embedding of a direct summand and the composition of all arrows is just the precomposition with n-fold twist by Frobenius homomorphism. Now we can use "twist injectivity" to complete the proof.

□

2.2. **Preliminary calculations.** Recall that for any natural d the symbols I^d, Λ^d, S^d and Γ^d denote correspondingly the kth tensor, exterior, symmetric and divided power functors. Let us recall also de Rham and Koszul sequences which relate exterior and symmetric power functors and which served as a main tool in the paper [FLS] for performing $Ext_{\mathcal{F}}(I,.)$ calculations with coefficients in these functors. But as it was demonstrated in, [FS] de Rham and Koszul sequences can be used equally well in the category $\mathcal{P}$ for calculating $\mathrm{Ext}_{\mathcal{P}}(I^{(t)},.)$ with coefficients in functors of degree p^t.

Koszul sequence K_k is a sequence of functors of the form

$$0 \to \Lambda^k \to \Lambda^{k-1}\otimes S^1 \to \Lambda^{k-2}\otimes S^2 \to ... \to S^k \to 0$$

This sequence is exact for any natural k. De Rham sequence R_k is defined as:

$$0 \to S^k \to \Lambda^1\otimes S^{k-1} \to \Lambda^2\otimes S^{k-2} \to ... \to \Lambda^k \to 0$$

It is exact for k not being divisible by p and $H^*(R_{p^t}) = R_{p^{t-1}}$ in the category $\mathcal{F}$ and $H^*(R_{p^t}) = R^{(1)}_{p^{t-1}}$ in $\mathcal{P}$. The usefulness of these sequences for our purposes comes from the fact that all functors in them but the first and the last are diagonalizable and hence give trivial groups after applying to them $\mathrm{Ext}_{\mathcal{F}}(I,.)$ and $\mathrm{Ext}_{\mathcal{P}}(I^{(t)},.)$ functors. Using this observation we see that the Koszul sequence gives us a direct shift isomorphism of $\mathrm{Ext}_{\mathcal{F}}(I,\Lambda^k)$ and $\mathrm{Ext}_{\mathcal{F}}(I,S^k)$ by the hyperext spectral sequence. The same is true in the category $\mathcal{P}$ for the functors $\mathrm{Ext}_{\mathcal{P}}(I^{(t)},.)$. Below we show that these results remain true after precomposing with functors of small degree.

Theorem 2.2. *Assume that* $F \in \mathcal{P}_i$, $1 < i < p$, $j \geq 0$ *and* $0 < k < p$. *Then*

$$\mathrm{Ext}^j_{\mathcal{P}}(F^{(1)}, (\Lambda^k \otimes S^{p-k}) \circ F) = 0.$$

Proof: Assume first that $F = I^i$. By our assumption on k both functors Λ^k and S^{p-k} are direct summands in I^k and I^{p-k} correspondingly. Hence our theorem follows directly from the obvious fact that

(2.2.1)

$$\mathrm{Ext}^*_{\mathcal{P}}(I^{i(1)}, I^p \circ I^i) = 0.$$

Our assumption on i implies that the category $\mathcal{P}_i$ is semi-simple and equivalent to the category of $F_p[\Sigma_i]$-modules. This implies that every functor F is a sum of simple functors and every simple functor is a direct summand in I^i. Using our assumption on k as previously, for the proof of our theorem it is enough to show that

$$\mathrm{Ext}^j_{\mathcal{P}}(F^{(1)}, G_1 \otimes ... \otimes G_p) = 0$$

for simple functors $F, G_1, ..., G_p \in \mathcal{P}_i$. But this follows directly from 2.2.1 because F is a direct summand in I^i and $G_1 \otimes ... \otimes G_p$ is a direct summand in $I^p \circ I^i$.

□

2.3. **The groups** $\mathrm{Ext}^*_{\mathcal{P}}(F^{(1)}, S^p \circ F)$ **and** $\mathrm{Ext}^*_{\mathcal{P}}(F^{(1)}, \Lambda^p \circ F)$**.**

Using Koszul and de Rham sequences we give full calculations of the Ext-groups of interest if $F \in \mathcal{P}_i$ and $i < p$. In the future we will write R for the de Rham sequence R_p and K for K_p. We know that having a complex of functors C^* we have two spectral sequences for calculating hyperext groups $\mathrm{Ext}^*_{\mathcal{P}}(F, C^*)$. We will use always cohomological notation and our spectral sequences will have rth differential of bidegree $(r, 1-r)$. We will call as the first spectral sequence this one, in which we first calculate cohomology in the direction of C^*. We will denote it ${}_1E^{m,n}_*$ while the second will be denoted ${}_2E^{m,n}_*$. If necessary we will decorate these spectral sequences with coefficients.

Let us start from recalling how the sequences R and K and the knowledge of $\mathrm{Ext}^*_{\mathcal{P}}(I^{(1)}, I^{(1)})$ give the calculation of $\mathrm{Ext}^*_{\mathcal{P}}(I^{(1)}, \Lambda^p)$ and the full understanding of differentials in ${}_1E^{m,n}_*(I^{(1)}, R)$, ${}_2E^{m,n}_*(I^{(1)}, R)$ and ${}_2E^{m,n}_*(I^{(1)}, K)$. Note that ${}_1E^{m,n}_*(I^{(1)}, R)$ has only two nontrivial rows for $n = 0, 1$ and they are equal to the graded group $\mathrm{Ext}^*_{\mathcal{P}}(I^{(1)}, I^{(1)}) = F_p[y]/y^p$, usually denoted as A^*. The generator y is placed in degree 2 and the identification is compatible with ring structures of both sides. Because S^p is injective we know that $\mathrm{Ext}^*_{\mathcal{P}}(I^{(1)}, S^p)$ consists of one copy of F_p in degree 0. From ${}_2E^{m,n}_*(I^{(1)}, K)$, which converges to 0, we know that $\mathrm{Ext}^*_{\mathcal{P}}(I^{(1)}, \Lambda^p)$ is nontrivial in degree $p-1$ and $\mathrm{Ext}^{p-1}_{\mathcal{P}}(I^{(1)}, \Lambda^p) = F_p$. From this we know that all differentials in ${}_2E^{m,n}_*(I^{(1)}, R)$ are trivial and

$$d_2 :_1 E^{0,1}_*(I^{(1)}, R) = F_p \to F_p =_1 E^{2,0}_*(I^{(1)}, R)$$

is an isomorphism. Moreover the differentials in our spectral sequences are homomorphisms of A^*-modules. This gives us full understanding of the differentials in ${}_1E_*^{m,n}(I^{(1)}, R)$.

Now we can move towards new calculations. Let $F \in \mathcal{P}_i$ for $i < p$ and let F^* denote the graded ring $\mathrm{Ext}^*_{\mathcal{P}}(F^{(1)}, F^{(1)})$. As previously the spectral sequence ${}_1E_*^{m,n}(F^{(1)}, R \circ F)$ has only two nontrivial rows for $n = 0, 1$ and this rows are isomorphic to $F^* = \mathrm{Ext}^*_{\mathcal{P}}(F^{(1)}, F^{(1)})$. Moreover the differentials are module maps over F^*.

Proposition 2.3. *There exists a nontrivial element $s \in F^2$ such that the only nontrivial differential $d_2 : {}_1E_2^{p,1}(F^{(1)}, R \circ F) \to_1 E_2^{p+2,0}(F^{(1)}, R \circ F)$ is the same as the map $F^* \to F^*$ induced by multiplication with s.*

Proof: Precomposition with F induces a map of hyperext groups

$$\mathrm{Ext}^*_{\mathcal{P}}(I^{(1)}, R) \to \mathrm{Ext}^*_{\mathcal{P}}(F^{(1)}, R \circ F)$$

and a map of spectral sequences

$$(*) \quad {}_1E_*^{m,n}(I^{(1)}, R) \to \ {}_1E_*^{m,n}(F^{(1)}, R \circ F)$$

which commutes with differentials. By Theorem 2.1 this map is injective on the second table.

The map

$$d_2 : \mathrm{Hom}(F^{(1)}, F^{(1)}) =_1 E_2^{0,1}(F^{(1)}, R \circ F) \to_1 E_2^{2,0}(F^{(1)}, R \circ F) = F^2$$

sends identity to a certain element $s \in F^2$. By the multiplicativity of the differential, d_2 is the same as multiplication by s. The element $s \neq 0$ because by (*) above and injectivity of the map of spectral sequences we know that it is equal to the nontrivial element coming from $y \in A^2$. □

Observe that in the case when $F = S^i$ we can be more specific. By [FFSS] we know that $\mathrm{Ext}^*_{\mathcal{P}}(S^{i(1)}, S^{i(1)}) = \Gamma^i(A)$. The second grade of $\Gamma^i(A)$ is equal to F_p spanned by s_1 - the first symmetric function. Hence the map

$$d_2 : \ {}_1E_2^{0,1}(S^{i(1)}, R \circ S^i) = F_p \to F_p = \ {}_1E_2^{2,0}(S^{i(1)}, R \circ S^i)$$

is an isomorphism and sends $1 \in F_p$ to the nontrivial multiple of s_1.

All higher differentials in ${}_1E_*^{m,n}(F^{(1)}, R \circ F)$ are trivial by degree reason. Hence the third table gives us the calculation of the hyperext groups $\mathrm{Ext}^*_{\mathcal{P}}(F^{(1)}, R \circ F)$. Let us write for shortness $d : F^* \to F^*$ instead of $d_2 : \ {}_1E_2^{m,1}(F^{(1)}, R \circ F) \to \ {}_1E_2^{m+2,0}(F^{(1)}, R \circ F)$. We know that $d(z) = s \cdot z$. This implies that $im(d) = s \cdot F^*$. The following proposition is a crucial technical step towards final calculations.

Proposition 2.4. *For $i < p$ the groups $\mathrm{Ext}^*_{\mathcal{P}}(I^{i(1)}, S^p \circ I^i)$ are nontrivial only in even degrees.*

Proof: The calculation is based on the Cauchy formula, proved in [ABW, Section III]. It implies that $S^p(V \otimes W)$ has filtration with quotients $S_\lambda(V) \otimes S_\lambda(W)$, where λ runs through all diagrams of degree p. Recall (following [J]) that Schur functors related to hooks $\{H_i\}_{i=0,1,\dots,p}$ fit into exact sequences

2.4.1

$$0 \to S_{H_j} \to S^j \otimes \Lambda^{p-j} \to S_{H_{j+1}} \to 0$$

where $H_p = S^p$.

We write $I^i = I \otimes I^{i-1}$. It is enough to show that for all diagrams λ of degree p the groups $\mathrm{Ext}^*_{\mathcal{P}}(I^{i(1)}, S_\lambda \circ I \otimes S_\lambda \circ I^{i-1})$ are concentrated in even degrees only.

Let π denote the functor taking two vector spaces to its sum. By adjointness of π and the diagonal embedding we know that

$$\mathrm{Ext}^*_{\mathcal{P}}(I^{i(1)}, S_\lambda \circ I \otimes S_\lambda \circ I^{i-1}) = \mathrm{Ext}^*_{bi-\mathcal{P}}(I^{i(1)} \circ \pi, S_\lambda \circ I \boxtimes S_\lambda \circ I^{i-1}).$$

The right-hand side decomposes into a sum of terms

$$\mathrm{Ext}^*_{\mathcal{P}}(I^{k(1)}, S_\lambda \circ I) \otimes \mathrm{Ext}^*_{\mathcal{P}}((I^{(i-k)(1)}, S_\lambda \circ I^{i-1})$$

By degree reasons such term can be nontrivial only when $k = 1$. The group $\mathrm{Ext}^*_{\mathcal{P}}(I^{(1)}, S_\lambda)$ is non trivial only for λ representing a hook. Hence we will assume for the rest of the proof of 2.4 that we are working only with hooks of degree p.

For $i = 2$ we get two copies of

$$\mathrm{Ext}^*_{\mathcal{P}}(I^{(1)}, S_\lambda) \otimes \mathrm{Ext}^*_{\mathcal{P}}(I^{(1)}, S_\lambda)$$

The functor S^p is injective, $\mathrm{Ext}^*_{\mathcal{P}}(I^{(1)}, S^p) = F_p$ in dimension 0. From this and 2.4.1 we get that for any j, $\mathrm{Ext}^*_{\mathcal{P}}(I^{(1)}, S_{H_j})$ is non zero in degree $p - j$ only and hence

$$\mathrm{Ext}^*_{\mathcal{P}}(I^{(1)}, S_{H_j}) \otimes \mathrm{Ext}^*_{\mathcal{P}}(I^{(1)}, S_{H_j})$$

is in even degree $2(p - j)$.

Now we can proceed by induction with respect to i. The groups

$$\mathrm{Ext}^*_{\mathcal{P}}(I^{(i-1)(1)}, (S^j \otimes \Lambda^{p-j}) \circ I^{(i-1)})$$

are trivial for $j \neq 0, p$ by Theorem 2.2. The exact sequences 2.4.1 remain exact after precomposition with $I^{(i-1)}$. From this we get immediately that for any j the parity of

$$\mathrm{Ext}^*_{\mathcal{P}}(I^{(1)}, S_{H_j})$$

is the same as of $\mathrm{Ext}^*_{\mathcal{P}}(I^{(i-1)(1)}, S_{H_j} \circ I^{i-1})$. This finishes the proof. □

Proposition 2.4 implies that for any $i < p$ and $F \in \mathcal{P}_i$ the groups $\mathrm{Ext}^*_{\mathcal{P}}(F^{i(1)}, S^p \circ F)$ are nontrivial in even degrees only. From Theorem 2.2 we know that ${}_1E^{m,n}_\infty(F^{(1)}, R \circ F)$ has only two non-zero rows, the 0th and the first. Moreover we know that ${}_1E^{*,0}_\infty(F^{(1)}, R \circ F) = F^*/s \cdot F^*$ and ${}_1E^{*,1}_\infty(F^{(1)}, R \circ F)$ have the same dimension.

Theorem 2.5. *We have the following formulas in the category of F^*-modules:*

- $Ext^*_{\mathcal{P}}(F^{(1)}, S^p \circ F) = F^*/s \cdot F^*$.
- $Ext^*_{\mathcal{P}}(F^{(1)}, \Lambda^p \circ F) = ker(s : F^* \to F^*)[-p+1]$.

Proof: Observe that even dimensional classes in the hyperext graded group $\mathrm{Ext}^*_{\mathcal{P}}(F^{(1)}, R \circ F)$ are contained in the 0 row of the ∞-table of the first spectral sequence and the odd classes are in the first row. By 2.3 we know that this 0-row is a cyclic F^*-module isomorphic to $F^*/s \cdot F^*$. It is generated by an element α of degree 0 coming from the $id \in \mathrm{Hom}_{\mathcal{P}}(F^{(1)}, F^{(1)})$ mapped to $\mathrm{Ext}^0_{\mathcal{P}}(F^{(1)}, R \circ F)$ by the Frobenius morphism $F^{(1)} \to S^p \circ F$. Similarly the first row is isomorphic to $ker(s : F^* \to F^*)$. Both rows have the same total dimension over $\mathbf{k}$.

By 2.2 we know that in the second spectral sequence ${}_2E^{m,n}_*(F^{(1)}, R \circ F)$ we have only two non zero columns for $m = 0, p$. The zero column consists of $Ext^*_{\mathcal{P}}(F^{(1)}, S^p \circ F)$ while the pth is equal to $\mathrm{Ext}^*_{\mathcal{P}}(F^{(1)}, \Lambda^p \circ F)$. Both spectral sequences converge to the same groups. From the Koszul spectral sequence we know that total dimension over $\mathbf{k}$ of $\mathrm{Ext}^*_{\mathcal{P}}(F^{(1)}, S^p \circ F)$ is the same as the dimension of $\mathrm{Ext}^*_{\mathcal{P}}(F^{(1)}, \Lambda^p \circ F)$. The only nontrivial differential in the Koszul spectral sequence is an isomorphism of F^*-modules. From this it follows that $\mathrm{Ext}^*_{\mathcal{P}}(F^{(1)}, \Lambda^p \circ F)$ contains a cyclic submodule generated by some class $\beta \in \mathrm{Ext}^{p-1}_{\mathcal{P}}(F^{(1)}, \Lambda^p \circ F)$. This class β is taken by the differential in the Koszul spectral sequence to the class α described above.

Proposition 2.4 shows that the zero row

$${}_1E^{\infty}_{*,0}(F^{(1)}, R \circ F) = F^*/s \cdot F^*$$

gives classes in $\mathrm{Ext}^*_{\mathcal{P}}(F^{(1)}, S^p \circ F)$ and the first row of this spectral sequence is contained in $\mathrm{Ext}^*_{\mathcal{P}}(F^{(1)}, \Lambda^p \circ F)$. It means that $\mathrm{Ext}^*_{\mathcal{P}}(F^{(1)}, S^p \circ F)$ contains a cyclic F^*-module $F^*/s \cdot F^*$ while $ker(s : F^* \to F^*)[-p+1]$ is contained in $\mathrm{Ext}^*_{\mathcal{P}}(F^{(1)}, \Lambda^p \circ F)$.

We have only to show that this way we get all classes in $\mathrm{Ext}^*_{\mathcal{P}}(F^{(1)}, S^p \circ F)$ and $\mathrm{Ext}^*_{\mathcal{P}}(F^{(1)}, \Lambda^p \circ F)$.

Assume that $\gamma \in_2 E^{m,n}_2(F^{(1)}, R \circ F)$ is an additional class of highest total degree j. It has to be killed by the only one possibly nontrivial differential. We can assume that j is finite because the cohomological dimension of the category $\mathcal{P}_{pi}$ is finite. We have two possibilities:

1. - j is even. Then it appears in the 0-column of ${}_2E^{m,n}_*(F^{(1)}, R \circ F)$. It cannot survive to infinity hence the only possibly nontrivial differential has to map it to some class γ' in the pth column. But then the total degree of γ' is $j+1$ which contradicts the definition of j.

2. - j is odd. This implies that $\gamma \in \mathrm{Ext}^{j-p+1}_{\mathcal{P}}(F^{(1)}, \Lambda^p \circ F)$. This implies that above the grade $j-p+1$ all classes in $\mathrm{Ext}^*_{\mathcal{P}}(F^{(1)}, \Lambda^p \circ F)$ belong to the submodule generated by β. From the Koszul spectral sequence we get immediately that for $k > j - 2p + 2$ all classes in $\mathrm{Ext}^k_{\mathcal{P}}(F^{(1)}, S^p \circ F)$ are

contained in the submodule generated by α. These classes have to survive to ∞. On the other hand the class which possibly kills γ is contained in $\mathrm{Ext}^j_{\mathcal{P}}(F^{(1)}, S^p \circ F)$ and this is impossible. $\Box$

Let us finish this subsection with two observations:

1. By construction, the first isomorphism from 2.5 is induced by the Frobenius morphism $f : F^{(1)} \to S^p \circ F$.

2. For $F = S^i$ the class β from the proof of 2.5 presents a shifted down by $p-1$ version of s_1^{p-1}. This implies that $ker(s_1 : \Gamma^i(A) \to \Gamma^i(A))$ is a cyclic submodule of $\Gamma^i(A)$ generated by s_1^{p-1}.

2.4. **On** $\mathrm{Ext}^*_{\mathcal{P}}(S^p \circ F, S^p \circ F)$ **for** $F \in \mathcal{P}_i, 1 < i < p$**.** We start our calculations from the very special case $F = S^2$. The theorem below should be treated as a baby version of the main computation.

Theorem 2.6. *The groups* $\mathrm{Ext}^i_{\mathcal{P}}(S^p \circ S^2, S^p \circ S^2)$ *are* 1*-dimensional over* $\mathbf{k}$ *in dimensions* $4k$ *and* $4k-1$ *for* $0 < i \leq (2p-2)$ *and are trivial in other degrees.*

Proof: As a tool we will use again the spectral sequences related to hyperext groups with coefficients in $R \circ S^2$ and $K \circ S^2$. But now we will use complexes of functors as covariant coefficients. At the beginning consider the two spectral sequences related to de Rham sequence precomposed with S^2. The first one ${}_1E^{m,n}_\infty(R \circ S^2, S^p \circ S^2)$ has only two nonzero rows, the $(p-1)$ and the pth. They are equal to $\mathrm{Ext}^*_{\mathcal{P}}(S^{2(1)}, S^p \circ S^2) = \Gamma^2(A)/s_1 \cdot \Gamma^2(A)$ by 2.5. By the Newton formulas we easily get that in positive degrees they are spanned by the powers s_2^i for $i \leq (p-1)/2$ of the second symmetric function s_2, which lies in degree 4. This implies that all differentials d_i in considered spectral sequence are trivial for $1 < i$ by degree reasons. Of course we know that $\mathrm{Ext}^0_{\mathcal{P}}(S^{2(1)}, S^p \circ S^2) = \mathbf{k}$. Hence we completely know the hyperext groups $\mathrm{Ext}^*_{\mathcal{P}}(R_p \circ S^2, S^p \circ S^2)$. They are equal to $\mathbf{k}$ in dimensions $p + 4k$ and $p + 4k - 1$ for $k = 0, 1, ..., (p-1)/2$.

In the second spectral sequence ${}_2E^{m,n}_\infty(R \circ S^2, S^p \circ S^2)$ we have two nonzero columns, the 0 and the pth. For $0 < j < p$ the functors $(S^j \otimes \Lambda^{(p-j)}) \circ S^2$ are direct summands of $(S^1)^{\otimes 2p}$ and hence are both projective and injective. This implies that other columns in ${}_2E^{m,n}_\infty(R \circ S^2, S^p \circ S^2)$ are trivial above dimension 0.

Assume that a certain class of degree q higher than $p-1$ survives to ∞ in the 0 column

$$ {}_2E^{0,q}_\infty(R \circ S^2, S^p \circ S^2) = \mathrm{Ext}^q_{\mathcal{P}}(\Lambda^p \circ S^2, S^p \circ S^2) $$

This implies that in $\mathrm{Ext}^*_{\mathcal{P}}(S^p \circ S^2, S^p \circ S^2)$ we have a nontrivial element in degree $q + p - 1$ (which is higher than $2p-2$) by the standard spectral sequence argument for hyperexts, using Koszul complex instead of de Rham one. But this implies that there is a nontrivial class in ${}_2E_\infty^{q+p-1,p}(R_p \circ S^2, S^p \circ S^2)$ which cannot survive to ∞ and has to be killed by an element from $\mathrm{Ext}_{\mathcal{P}}^{q+2p-2}(\Lambda^p \circ S^2, S^p \circ S^2)$. This observation leads immediately to the contradiction with the finite cohomological dimension of the category $\mathcal{P}_{2p}$. □

It is worth to underline that our computations show that the groups $Ext_{\mathcal{P}}^*(\Lambda^p \circ S^2, S^p \circ S^2)$ are trivial in dimensions greater than $p-1$. In dimensions below $p-1$ they are closely related to the cohomology of the complex $\mathrm{Hom}_{\mathcal{P}}(R \circ S^2, S^p \circ S^2)$.

Let L be the cokernel of the natural map $f : I^{(1)} \to S^p$. Equally well we can define it as a kernel of the dual map $f^* : \Gamma^p \to I^{(1)}$. Remember that the natural number i satisfies $2 < i < p$ and $F \in \mathcal{P}_i$. The following theorem is crucial for our computations:

Theorem 2.7. *The maps induced on cohomology by the Frobenius morphism have the following properties:*

- *The map f induces an epimorphism* $\mathrm{Ext}_{\mathcal{P}}^*(S^p \circ F, S^p \circ F) \to \mathrm{Ext}_{\mathcal{P}}^*(I^{(1)} \circ F, S^p \circ F)$.
- *The map f^* induces a zero map* $\mathrm{Ext}_{\mathcal{P}}^t(I^{(1)} \circ F, S^p \circ F) \to \mathrm{Ext}_{\mathcal{P}}^t(\Gamma^p \circ F, S^p \circ F)$ *for any $t > 0$. In dimension 0, f^* induces an embedding*

$$\mathrm{Ext}_{\mathcal{P}}^0(I^{(1)} \circ F, S^p \circ F) \hookrightarrow \mathrm{Ext}_{\mathcal{P}}^0(\Gamma^p \circ F, S^p \circ F).$$

Proof: As a tool we will use the double complex $A^{m,n}$ which vertically consists of pieces of the de Rham sequence and horizontally pieces of the Koszul sequence:

$$\begin{array}{ccccccccc}
S^p & & & & & & & & \\
\downarrow & & & & & & & & \\
S^{p-1} \otimes S^1 & \to & S^p & \to & 0 & & & & \\
\downarrow & & \downarrow & & & & & & \\
\vdots & \vdots & \vdots & \vdots & & & & & \\
\downarrow & & \downarrow & & & & & & \\
S^2 \otimes \Lambda^{p-2} & \to & S^3 \otimes \Lambda^{p-3} & \to & \dots & \to & S^p & \to & 0 \\
\downarrow & & \downarrow & & & & \downarrow & & \\
S^1 \otimes \Lambda^{p-1} & \to & S^2 \otimes \Lambda^{p-2} & \to & \dots & \to & S^{p-1} \otimes S^1 & \to & S^p \to 0
\end{array}$$

By [J, section 3] we know that the complex obtained from the above double complex gives us an injective resolution of $I^{(1)}$. After precomposing it with F we get a cohomological resolution J^* of $F^{(1)}$, which of course in not injective any more. We have $J^0 = S^p \circ F$ and generally

$$J^k = \bigoplus_{j=0}^{\infty} (S^{p-k+2j} \otimes \Lambda^{k-2j}) \circ F$$

Now we apply the functor $\mathrm{Hom}_{\mathcal{P}}(-, S^p \circ F)$ to it and calculate two spectral sequences coming from the double complex structure. One is simple (J^* has cohomology only in dimension 0) and tells us that both spectral sequences converge to $\mathrm{Ext}^*_{\mathcal{P}}(F^{(1)}, S^p \circ F)$. Observe that for $0 < k < p$ all functors $(S^k \otimes \Lambda^{p-k}) \circ F$ are direct summands in $(S^1)^{\otimes pi}$ and hence are projective and injective. It means that in the second spectral sequence we have nontrivial 0-row and p columns consisting of $\mathrm{Ext}^*_{\mathcal{P}}(S^p \circ F, S^p \circ F)$ separated by columns with zeros above dimension 0. Let $E_2^{m,n}$ be the second table of our second spectral sequence. Then the table has $2p-1$ nontrivial columns (numbered from 0 to $2p-2$), $d_2 : E_2^{m,n} \to E_2^{m+2,n-1}$ and we have:

$$E_2^{m,n} = \mathrm{Ext}^t_{\mathcal{P}}(S^p \circ F, S^p \circ F) \text{ for even m}$$

and

$$E_2^{m,n} = 0 \text{ for odd m and n} > 0.$$

By [J, section 1] we know that there is an exact sequence:

$$0 \to I^{(1)} \to S_0 \to S_1 \to \cdots \to S_{p-1} \to 0$$

where $S_0, \ldots, S_{p-1}$ are Schur functors ($S_0 = S^p$). We will denote as Z^* the above complex without $I^{(1)}$ after precomposing it with $F^{(1)}$. The Schur functors are defined as the kernels of the Koszul differential

$$S_k = ker(S^{p-k} \otimes \Lambda^k \to S^{p-k+1} \otimes \Lambda^{k-1})$$

It means that we have a map of exact sequences

$$\begin{array}{ccccc} 0 & \to & F^{(1)} & \to & Z^* \\ & & id \downarrow & & \downarrow \\ 0 & \to & F^{(1)} & \to & J^* \end{array}$$

which starts of course from the diagram

$$\begin{array}{ccccc} 0 & \to & F^{(1)} & \to & S^p \circ F \\ & & id \downarrow & & id \downarrow \\ 0 & \to & F^{(1)} & \to & S^p \circ F \end{array}$$

The map $Z^* \to J^*$ is an equivalence of complexes because it extends the $id : F^{(1)} \to F^{(1)}$. Hence it should give us the isomorphism on the hyperext groups

$$\mathrm{Ext}^*_{\mathcal{P}}(J^*, S^p \circ F) \to \mathrm{Ext}^*_{\mathcal{P}}(Z^*, S^p \circ F).$$

On the other hand, on the level of spectral sequences of the second type this map is induced on every column, except the zero one, by the embeddings

$$S_k \circ F \to (S^{p-k} \otimes \Lambda^k) \circ F$$

and has to be trivial in degrees higher than 0 by obvious reasons. But this implies, that the isomorphism

$$\mathrm{Ext}^*_{\mathcal{P}}(J^*, S^p \circ F) \to \mathrm{Ext}^*_{\mathcal{P}}(Z^*, S^p \circ F)$$

has to be realized by the identity map of the last columns (indexed $2p-1$). This implies the first statement of our theorem.

For the second we have to proceed in a dual manner. Recall that for finite functors G and H the Kuhn's duality satisfies

$$D(G \circ H) = DG \circ DH.$$

Moreover we know that $DS^p = \Gamma^p$ and $D(S^k \otimes \Lambda^{p-k}) = S^k \otimes \Lambda^{p-k} = \Gamma^k \otimes \Lambda^{p-k}$ for $0 < k < p$. From this we get that

$$DJ^k = \bigoplus_{j=0}^{\infty} (\Gamma^{p-k+2j} \otimes \Lambda^{k-2j}) \circ F$$

and DJ^* is a resolution of $F^{(1)}$ because F and DF are isomorphic. The last statement follows from the fact that F is a direct sum of simple objects, compare [K1, Theorem 9.1]. As previously we apply the functor $\mathrm{Hom}_{\mathcal{P}}(-, S^p \circ F)$ to it and study two spectral sequences coming from double complex structure. One of them tells us that both of them converge to $\mathrm{Ext}^*_{\mathcal{P}}(F^{(1)}, S^p \circ F)$. The second one $DE^{m,n}_*$ has $2p-1$ nontrivial columns numbered from 0 to $2p-2$, $d_2 : DE^{m,n}_2 \to DE^{m+2,n-1}_2$ and we have:

$$DE^{m,n}_2 = \mathrm{Ext}^n_{\mathcal{P}}(\Gamma^p \circ F, S^p \circ F) \quad \text{for even m}$$

and

$$DE^{m,n}_2 = 0 \quad \text{for odd m and n} > 0.$$

But for the proof of the second statement of the theorem we need some additional feature of the bicomplex $A^{*,*}$. Observe that the diagonal identities $id : S^k \otimes \Lambda^{p-k} \to S^k \otimes \Lambda^{p-k}$ commute with vertical and horizontal differentials and give us degree 2 map of complexes $x : J^* \to J^{*+2}$ and $Dx : DJ^* \to DJ^{*-2}$. Of course x acts also on spectral sequences, commuting with all differentials. We will denote this action by the same letter x in the future. The map

$$x : DE^{m,n}_2 \to DE^{m+2,n}_2$$

is equal to the identity on even columns because it is induced by $id : \Gamma^p \circ F \to \Gamma^p \circ F$.

Assume that the second statement of the theorem is not true. In such case there should be an element $\alpha \in DE^{0,n}_2$ for some $n > 0$, which is nontrivial and survives to infinity. But then $x \cdot \alpha$ also survives to infinity because x commutes with differentials and hence sends cocycles to cocycles. It means that x acts non trivially on $DE^{m,n}_\infty$ and hence is nontrivial on the limit $\mathrm{Ext}^*_{\mathcal{P}}(F^{(1)}, S^p \circ F)$. But this is impossible by the lemma 2.8 below. The calculation in dimension 0 easily follows from the long exact sequence related to f^*.

Lemma 2.8. *The operator* $x : DE_\infty^{m,n} \to DE_\infty^{m+2,n}$ *is trivial for* $n > 0$.

Proof. Let $IE_*^{m,n}$ be the spectral sequence obtained from DJ^* after applying to it $\mathrm{Hom}_{\mathcal{P}}(-, F^{(1)})$. Then $IE_*^{s,t}$ converges to $\mathrm{Ext}^*_{\mathcal{P}}(F^{(1)}, F^{(1)})$, the map $f : F^{(1)} \to S^p \circ F$, applied to the second variable, induces a map of spectral sequences $\bar{f} : IE_*^{m,n} \to DE_*^{m,n}$. Operator x acts also on $IE_*^{m,n}$ and $\bar{f}$ is equivariant with respect to this action. On the limit of spectral sequences $\bar{f}$ is equal to the natural quotient map

$$F^* = \mathrm{Ext}^*_{\mathcal{P}}(F^{(1)}, F^{(1)}) \to \mathrm{Ext}^*_{\mathcal{P}}(F^{(1)}, S^p \circ F) = F^*/s \cdot F^*.$$

The operator $x : F^* \to F^*$ is of degree 2 and its action commutes with left multiplication. By [J, Lemma 3.4] the operator x acts on $\mathrm{Ext}^*_{\mathcal{P}}(I^{(1)}, I^{(1)}) = F_p[y]/y^p$ as multiplication by y. This implies that x acts on F^* as multiplication by s and hence acts trivially on $F^*/s \cdot F^*$.

Theorem 2.9. *Let* $F \in P_i$ *for* $0 < i < p$ *and* $X = \mathrm{Ext}^*_{\mathcal{P}}(F^{(1)}, S^p \circ F)$. *Additively the graded* F_p*-vector space* $\mathrm{Ext}^*_{\mathcal{P}}(S^p \circ F, S^p \circ F)$ *in degrees above* 0 *is isomorphic to the positively graded part of the graded vector space* $X \otimes \Lambda(a) \otimes S(b)$ *where the exterior generator* a *has degree* -1 *and the polynomial generator* b *has degree* $2 - 2p$.

Proof: We will construct a finite filtration $\mathrm{Ext}^*_{\mathcal{P}}(S^p \circ F, S^p \circ F) = A_0 \supset A_1 \supset A_2 \supset ...$ with quotients isomorphic to the quotients of the filtration of $X \otimes \Lambda(a) \otimes S(b)$ defined by the powers of the polynomial generator b. We have two main tools. The first is Theorem 2.7 above. The second comes from the exact sequence

$$0 \to \Gamma^p \circ F \to (\Gamma^{p-1} \otimes \Lambda^1) \circ F \to \cdots \to (S^{p-1} \otimes \Lambda^1) \circ F \to S^p \circ F \to 0$$

which is obtained by gluing at Λ^p the Koszul exact sequence with its dual and precomposing all terms with F. We apply $\mathrm{Hom}_{\mathcal{P}}(-, S^p \circ F)$ to it and obtain a spectral sequence converging to 0, which has $2p$ nontrivial columns numbered from 0 to $2p - 1$. The 0-column is equal to $\mathrm{Ext}^*_{\mathcal{P}}(S^p \circ F, S^p \circ F)$, the $2p - 1$ column is equal to $\mathrm{Ext}^*_{\mathcal{P}}(\Gamma^p \circ F, S^p \circ F)$ and all other columns are trivial above dimension 0. The differential d_{2p-1} gives us an isomorphism

$$d_{2p-1} : \mathrm{Ext}^{*+2p-2}_{\mathcal{P}}(S^p \circ F, S^p \circ F) \xrightarrow{\sim} \mathrm{Ext}^*_{\mathcal{P}}(\Gamma^p \circ F, S^p \circ F).$$

Theorem 2.7 gives us two families of short exact sequences:

$$0 \to \mathrm{Ext}^*_{\mathcal{P}}(L \circ F, S^p \circ F) \to \mathrm{Ext}^*_{\mathcal{P}}(S^p \circ F, S^p \circ F) \to$$
$$\to \mathrm{Ext}^*_{\mathcal{P}}(F^{(1)}, S^p \circ F) = X^* \to 0$$

$$0 \to \mathrm{Ext}^*_{\mathcal{P}}(\Gamma^p \circ F, S^p \circ F) \to \mathrm{Ext}^*_{\mathcal{P}}(L \circ F, S^p \circ F) \to$$
$$\to \mathrm{Ext}^{*+1}_{\mathcal{P}}(F^{(1)}, S^p \circ F) = X^{*+1} \to 0$$

which of course split over $\mathbf{k}$.

Denote $A_1 = \mathrm{Ext}^*_{\mathcal{P}}(\Gamma^p \circ F, S^p \circ F)$ and $A_0 = \mathrm{Ext}^*_{\mathcal{P}}(S^p \circ F, S^p \circ F)$. Then A_1 embeds into A_0 by composing first arrows in described exact sequences. Moreover it is obvious that $A_0/A_1 = X \otimes \Lambda(a)$.

WARNING: Below, when we talk about graded objects, we always think only about their non-negative parts.

The differential d_{2p-1} gives us embedding of $A_2 = A_1[2p-2]$ into A_1 and hence also into A_0. Observe that

$$A_1/A_2 \simeq (X \otimes \Lambda(a))[2p-2]$$

Denote $A_3 = A_2[2p-2] = A_1[4p-4]$. Then differential d_{2p-1} embeds it into A_2 and hence into A_0 as previously. Again we see that

$$A_2/A_3 \simeq (X \otimes \Lambda(a))[4p-4]$$

.

Now the proof of 2.9 is clear by induction. We define A_k as $A_{k-1}[2p-2]$ and calculate that

$$A_{k-1}/A_k \simeq (X \otimes \Lambda(a))[(k-1)(2p-2)].$$

This way we obtain the promised finite filtration $A_1 \supset A_2 \supset \dots$ of A_0 with quotients isomorphic to the quotients of the natural filtration of $X \otimes \Lambda(a) \otimes S(b)$ defined by the powers of b. The lengths of the filtration depends on p and i. □

Observe that the calculations from 2.9 are valid also in degree 0 if we divide $\mathrm{Ext}^0_{\mathcal{P}}(\Gamma^p \circ F, S^p \circ F)$ by the image of the last map in the zero row

$$\mathrm{Ext}^0_{\mathcal{P}}((\Gamma^{p-1} \otimes \Lambda^1) \circ F, S^p \circ F) \to \mathrm{Ext}^0_{\mathcal{P}}(\Gamma^p \circ F, S^p \circ F).$$

Question: Is it true that the formula $\mathrm{Ext}^*_{\mathcal{P}}(S^p \circ F, S^p \circ F) \simeq X \otimes \Lambda(a) \otimes S(b)$ holds also multiplicatively? We consider here X as a ring with the quotient ring structure $F^* \to X = F^*/s \cdot F^*$. This is true for $F = S^2$, see the next section.

3. Multiplicative structures

In this section we calculate the Yoneda ring structure on $\mathrm{Ext}^*_{\mathcal{P}}(S^p \circ S^2, S^p \circ S^2|)$. The situation in higher Ext degrees and that in the Hom groups is quite different and requires different methods. We study the former one in Subsection 3.1 and the latter one and their interplay in Subsection 3.3. We provide in Subsection 3.2 a brief exposition of the relevant material on Hecke algebras, which will be used in Subsection 3.3

3.1. **On** $\mathrm{Ext}^{>0}_{\mathcal{P}}(S^p \circ S^2, S^p \circ S^2)$.

Let B^* be a graded $\mathbf{k}$-algebra, which in positive degrees is equal to $\mathrm{Ext}^i_{\mathcal{P}}(S^p \circ S^2, S^p \circ S^2)$ and B^0 is one dimensional, spanned by the identity on $S^p \circ S^2$. Let B^{2*} be its even degree part. Theorem 2.6 tells us (compare observations after Theorem 2.5) that the map $f : S^{2(1)} \to S^p \circ S^2$ obtained from the Frobenius morphism induces an isomorphism

$$f^* : B^{2*} \to \mathrm{Ext}^{2*}_{\mathcal{P}}(S^{2(1)}, S^p \circ S^2).$$

Theorem 3.1. *The Yoneda multiplication in B^* is commutative and defined by:*

a. *The evenly graded algebra $B^{2\star}$ is isomorphic to $\mathbf{k}[t]/(t^{(p+1)/2})$, with the generator $t \in B^4$.*

b. *Multiplication by t from both sides induces an isomorphism $B^{(4k-1)} \to B^{(4(k+1)-1)}$.*

c. *Multiplication of odd dimensional classes is trivial.*

Proof: We will show that the multiplication in B^* is obtained directly from the multiplication in $Ext^*_{\mathcal{P}}(S^{2(1)}, S^{2(1)})$.

Proof of a. Let $a, b \in B^{2*}$. By the naturality of the Yoneda multiplication we know that $f^*(a \cdot b) = a \cdot f^*(b)$. Let $\bar{b}$ is the unique element satisfying $f^*(b) = \bar{b}$, for any $b \in B^{2*}$. By the second observation after theorem 2.5 in order to get multiplicative structure of B^{2*} it is enough to consider products of the type $a \cdot \bar{b}$. It means it is enough to consider Yoneda multiplication

$$\mathrm{Ext}^{2*}_{\mathcal{P}}(S^p \circ S^2, S^p \circ S^2) \otimes \mathrm{Ext}^{2*}_{\mathcal{P}}(S^{2(1)}, S^p \circ S^2) \to \mathrm{Ext}^{2*}_{\mathcal{P}}(S^{2(1)}, S^p \circ S^2).$$

Every class $\bar{b} \in \mathrm{Ext}^j_{\mathcal{P}}(S^{2(1)}, S^p \circ S^2)$ can be presented as $f_*(z)$ for a certain class $z \in \mathrm{Ext}^j_{\mathcal{P}}(S^{2(1)}, S^{2(1)})$ by theorem 2.5. Hence we have:

$$a \cdot \bar{b} = a \cdot f_*(z) = f^*(a) \cdot z = \bar{a} \cdot z$$

We know that $\bar{a} = f^*(w)$ for a certain $w \in \mathrm{Ext}^{2*}_{\mathcal{P}}(S^{2(1)}, S^{2(1)})$ (not uniquely) and as previously $f^*(w) \cdot z = f^*(w \cdot z)$. The classes $w, z \in \mathrm{Ext}^{2*}_{\mathcal{P}}(S^{2(1)}, S^{2(1)}) = \Gamma^2(A)$. Taking as w and z the correct powers of $s_2 \in \Gamma^2(A)$ gives us the desired result.

Proof of b. Let L be the cokernel of the Frobenius map $I^{(1)} \to S^p$. From 2.6 and the proof of a we know that the natural map induced by Frobenius

$$\mathrm{Ext}^*_{\mathcal{P}}(S^p \circ S^2, S^p \circ S^2) \to \mathrm{Ext}^*_{\mathcal{P}}(S^{2(1)}, S^p \circ S^2)$$

is an epimorphism of graded rings. From the long exact sequence of Ext-groups we know that its kernel is $\mathrm{Ext}^*_{\mathcal{P}}(L \circ S^2, S^p \circ S^2)$. This kernel consists of all odd dimensional classes in B^*.

Let Γ^p be the Kuhn dual of S^p. Let K be an exact sequence obtained as the composition of the Koszul exact sequence K_p with its Kuhn dual. We get this way an exact sequence of length $2p-2$ which starts from Γ^p and ends with S^p. Using spectral sequence for the hyperext groups $\mathrm{Ext}^*_{\mathcal{P}}(K \circ S^2, S^p \circ S^2)$ we immediately get that the groups $\mathrm{Ext}^*_{\mathcal{P}}(\Gamma^p \circ S^2, S^p \circ S^2)$ are trivial in positive degrees.

We have an exact sequence of functors

$$0 \to L \to \Gamma^p \to I^{(1)} \to 0.$$

The previous observation implies that the boundary map

$$\mathrm{Ext}^*_{\mathcal{P}}(L \circ S^2, S^p \circ S^2) \to \mathrm{Ext}^{*+1}_{\mathcal{P}}(S^{2(1)}, S^p \circ S^2)$$

is an isomorphism of left $\mathrm{Ext}^*_{\mathcal{P}}(S^p \circ S^2, S^p \circ S^2)$-modules. From the proof of **a.** we know what is this module structure on $Ext^*_{\mathcal{P}}(S^{2(1)}, S^p \circ S^2)$. This implies **b.** for the left multiplication.

For the right multiplication we will use several times the complex K described above and hyperext spectral sequences obtained from it. Using them we get immediately that in gradation j bigger than $2p-2$ we have:

- $\mathrm{Ext}^{j-2p+2}_{\mathcal{P}}(S^p \circ S^2, S^p \circ S^2)$ is isomorphic to $\mathrm{Ext}^j_{\mathcal{P}}(S^p \circ S^2, \Gamma^p \circ S^2)$ and this is an isomorphism of corresponding right $\mathrm{Ext}^*_{\mathcal{P}}(S^p \circ S^2, S^p \circ S^2)$-modules.
- $\mathrm{Ext}^{j-2p+2}_{\mathcal{P}}(S^p \circ S^2, S^{2(1)}) = \mathrm{Ext}^j_{\mathcal{P}}(S^{2(1)}, \Gamma^p \circ S^2)$ and hence $\mathrm{Ext}^{j-2p+2}_{\mathcal{P}}(S^p \circ S^2, S^{2(1)})$ are nontrivial only for even j.

From the exact sequence $0 \to S^{2(1)} \to S^p \circ S^2 \to L \circ S^2 \to 0$ and the calculation of $\mathrm{Ext}^*_{\mathcal{P}}(S^p \circ S^2, S^p \circ S^2)$ we have a boundary isomorphism $\mathrm{Ext}^j_{\mathcal{P}}(S^p \circ S^2, L^p \circ S^2) \to \mathrm{Ext}^{j+1}_{\mathcal{P}}(S^p \circ S^2, S^{2(1)})$ for $j > 2p-2$. This implies that for $j > 2p-2$ the groups $\mathrm{Ext}^j_{\mathcal{P}}(S^p \circ S^2, \Gamma^p \circ S^2) = \mathrm{Ext}^j_{\mathcal{P}}(S^p \circ S^2, S^{2(1)})$ for even j and $\mathrm{Ext}^j_{\mathcal{P}}(S^p \circ S^2, \Gamma^p \circ S^2) = \mathrm{Ext}^j_{\mathcal{P}}(S^p \circ S^2, L \circ S^2)$. Hence we can finish the proof as it was done for the left multiplication using $\mathrm{Ext}^*_{\mathcal{P}}(S^p \circ S^2, \Gamma^p \circ S^2)$ instead of $\mathrm{Ext}^*_{\mathcal{P}}(S^p \circ S^2, S^p \circ S^2)$.

Proof of c. The odd dimensional classes appear only in degrees $4k-1$ hence the result of multiplying two of them has degree $4l-2$ for a certain l. By theorem 2.6 it is equal to 0.

3.2. **On Hecke algebras and index.**
Hecke algebras appear in many contexts in algebra and allow various equivalent descriptions. The most common definition is the following. Given a pair of groups $H \subset G$, the Hecke algebra $He(G, H)$ as a vector space over $\mathbf{k}$ is spanned by the set of H double cosets in G, and the multiplication is given by the intricate combinatorial formula referred to as "convolution product". This is related to our situation via Proposition 1.4. The aim of this subsection is to clarify this connection and to gather some facts concerning Hecke algebras needed for description of Yoneda multiplication of plethysms. We do not claim any originality here, we just want to make our

paper more self-contained, while the relevant material is scattered in literature often with unnecessary restrictions on the characteristic of a ground ring.

We start with recalling various definitions of Hecke algebra, beginning with the most elementary one, which we mentioned earlier (we refer the reader for details eg. to [CR, Section 11D]). For a pair of finite groups $H \subset G$, let $A := \mathbb{Z}[\frac{1}{|H|}][G]$ stand for the group algebra of G with coefficients in $\mathbb{Z}[\frac{1}{|H|}]$. For a $H - H$ double coset x in G we put:

$$e_x := \frac{1}{|H|} \sum_{y \in HxH} y$$

and we put $He(G, H)$ to be the $\mathbb{Z}$-algebra generated by all e_x in A. Then one can show that $He(G, H)$ is a free $\mathbb{Z}$-module with basis $\{e_x\}_{x \in H\backslash G/H}$ and the multiplication is given by the formula:

$$e_x \cdot e_y = \sum_{z \in H\backslash G/H} \mu_{xyz} e_z,$$

where the structure constants are given by the formula:

$$\mu_{xyz} = \frac{|HxH \cap zHyH|}{|H|},$$

(see [CR, Prop. 11.34]). From this description one can immediately obtain an important $\mathbb{Z}$-character of $He(G, H)$ called index.

Definition/Proposition 3.2. *Let $a : A \longrightarrow \mathbb{Z}[\frac{1}{|H|}]$ be the standard augmentation on the group algebra (ie. $a(g) = 1$ for all $g \in G$). Then the image of composite:*

$$\mathrm{ind} : He(G, H) \subset A \xrightarrow{a} \mathbb{Z}[\frac{1}{|H|}]$$

is contained in $\mathbb{Z}$. Explicitly, we have: $ind(e_x) = \frac{|HxH|}{|H|}$.

The proof of both assertions follows from the fact that a double coset is a disjoint sum of single cosets. □

Let us also remark that the construction of $He(G, H)$ and *ind* can be repeated by taking scalars in any commutative ring $\mathbf{k}$ instead of $\mathbb{Z}$. We denote such a variant by $He(G, H)_{\mathbf{k}}$. In fac,t since $He(G, H)$ is a free $\mathbb{Z}$-module, we just have $He(G, H)_{\mathbf{k}} \simeq He(G, H) \otimes_{\mathbb{Z}} \mathbf{k}$. Of course in this case the index takes values in the image of $\mathbb{Z}$ in $\mathbf{k}$, although we will still denote it just by *ind*.

Now we shall discuss an alternative description of $He(G, H)_{\mathbf{k}}$, which is perhaps better motivated, yet somewhat less explicit.

Proposition 3.3. *There is an isomorphism of $\mathbf{k}$-algebras:*

$$\phi : \mathrm{Hom}_{G-\mathrm{mod}}(\mathbf{k}[G/H], \mathbf{k}[G/H]) \longrightarrow He(G, H)_{\mathbf{k}}$$

given by the formula $\phi(f) := f(1_G)$.

Proof follows from the fact that the map $f : \mathbf{k}[G/H] \longrightarrow \mathbf{k}[G/H]$ is G-equivariant if and only if it is constant on the double cosets. $\square$

Now let us express *ind* by using this description.

Proposition 3.4. *Any G-endomorphism $f : \mathbf{k}[G/H] \longrightarrow \mathbf{k}[G/H]$ preserves the kernel of the augmentation $a_{\mathbf{k}} : \mathbf{k}[G/H] \longrightarrow \mathbf{k}$, hence it induces the map $\bar{f} : \mathbf{k} \longrightarrow \mathbf{k}$. Thus we constructed the $\mathbf{k}$-algebra homomorphism:*

$$ind' : \mathrm{Hom}_{G-\mathrm{mod}}(\mathbf{k}[G/H], \mathbf{k}[G/H]) \longrightarrow \mathrm{Hom}_{\mathbf{k}-\mathrm{mod}}(\mathbf{k}, \mathbf{k}) \simeq \mathbf{k},$$

which satisfies: $ind \circ \phi = ind'$.

Proof immediately follows from the description of ϕ given in Proposition 3.3. $\square$

In order to make the formulas describing Yoneda multiplications given in Section 3.3 explicit, we will compute directly *ind* for $G = \Sigma_{2d}$, $H = \Sigma_2 \wr \Sigma_d$. We begin with a graph-like description of the elements of Σ_{2d} from which the bi-action of $\Sigma_2 \wr \Sigma_d$ can be easily read off. Namely, let $Gr_{bip}(d, 2; d, 2, \pm)$ denote the set of bipartite graphs (see Section 1.2) with the ends of edges decorated by a sign in such a way that to any vertex there is attached a one edge having plus and other having minus near our vertex. We emphasize that multiple edges are indistinguishable ie. $Gr_{bip}(1, 2; 1, 2, \pm)$ consists of the two (not four) decorated graphs. Now we shall establish bijection Δ between Σ_{2d} and $Gr_{bip}(d, 2; d, 2, \pm)$. For $\sigma \in \Sigma_{2d}$ we construct the graph $\Delta(\sigma)$ by the following procedure. For $i, j \in [d]$ we draw edge between i from the first copy of $[d]$ and j from the second copy of $[d]$ in the four cases: $\sigma(i) = j$, $\sigma(i) = d + j$, $\sigma(d + i) = j$, $\sigma(d + i) = d + j$ and we put accordingly the decorations: $(+, +), (+, -), (-, +), (-, -)$. Now we can very conveniently record the bi-action of $\Sigma_2 \wr \Sigma_d$. For the right action: $\Sigma_2^{\times d}$ changes decorations near vertices from the first copy of $[d]$ and Σ_d re-numerates the edges. Similarly, the left action changes decorations and remunerates edges in the second copy of $[d]$. Now, for example we can recover the graph-like descriptions from Section 1.2. Namely: taking double cosets with respect to $\Sigma_2^{\times d}$ forgets decorations, hence we get:

$$\Sigma_2^{\times d} \setminus \Sigma_{2d} / \Sigma_2^{\times d} \simeq \Sigma_2^{\times d} \setminus Gr_{bip}(d, 2; d, 2, \pm) / \Sigma_2^{\times d} \simeq Gr_{bip}(d, 2; d, 2). \tag{3.1}$$

In order to compute ind we shall describe the single cosets in graph-like terms. Analogously to Section 1.2 we have:

$$Gr_{bip}(d, 2; d, 2, \pm) / \Sigma_2 \wr \Sigma_d \simeq Gr(d, 2, \pm),$$

where $Gr(d, 2, \pm)$ stands for the set of decorated graphs with the set of vertices $[d]$, with each vertex of degree 2. We recall that since double edges in $Gr(d, 2; d, 2, \pm)$ are indistinguishable, $Gr(1, 2, \pm)$ consists of a single element and $Gr(2, 2, \pm)$ consists of two elements. The action of $\Sigma_2 \wr \Sigma_d$ on $Gr(d, 2, \pm)$ is the following: Σ_d re-numerates edges and $\Sigma_2^{\times d}$ changes decorations. We need to compute the cardinality of the set $\Sigma_2 \wr \Sigma_d \; \sigma \; \Sigma_2 \wr \Sigma_d$ for $\sigma \in \Sigma_{2d}$. By (3.1) it equals the index of the isotropy group of the $\Sigma_2 \wr \Sigma_d$-action on the graph $\Delta(\sigma)$ in $\Sigma_2 \wr \Sigma_d$. So, let determine these groups. Similarly to Section 1.2 again, the set of $\Sigma_2 \wr \Sigma_d$-orbits of graphs in $Gr(d,2,\pm)$ is in bijection with the set of Young diagrams of weight d, $\Lambda(d)$. So, let us λ be a Young diagram of weight d. We encode λ as $(\lambda_1^{a_1},\ldots,\lambda_k^{a_k})$ for $\lambda_1 > \ldots > \lambda_k$ and $\sum_i a_i\lambda_i = d$, ie. we record how many repetitions occur in λ. We define the graph $G_\lambda \in Gr(d,2,\pm)$ as having the cycles: $(1,\ldots,\lambda_1),\ldots,((a_1-1)\lambda_1+1,\ldots,a_1\lambda_1),\ldots,(d-\lambda_m+1,\ldots,d)$, with decorations $(+,-)$ along the edges $(j,j+1)$, with the standard cyclic convention. Let us first consider the case $\lambda=(d)$. Then the isotropy group is generated by the cycle $(1,\ldots,d)\in\Sigma_d$ and symmetry $\sigma^{\times d}\tau \in \Sigma_2\wr\Sigma_d$, where $\sigma\in\Sigma_2$ is the nontrivial element and $\tau\in\Sigma_d$ is given by $\tau(j)=d+1-j$. We shall denote this group by D_d, since for $d>2$ it is isomorphic to the group of isometries of the regular d-polygon. In particular we have $|D_d|=2d$ (also for $d\le 2$). For a general λ we encounter an additional phenomenon: if $a_i>1$ for some i, then G_λ has $a_i>1$ cycles of length i, and permuting them produces a copy of Σ_{a_i} inside the isotropy group. Therefore, the isotropy group of G_λ, which we shall denote by D_λ is the semidirect product:

$$D_\lambda := ((D_{\lambda_1})^{\times a_1}\times\ldots\times(D_{\lambda_k})^{\times a_k})\rtimes(\Sigma_{a_1}\times\ldots\times\Sigma_{a_k}).$$

Therefore, denoting by e_λ the basis element of $He(\Sigma_{2d}\Sigma_2\wr\Sigma_d)$ corresponding to the graph G_λ, we have:

$$ind(e_\lambda)=\frac{|\Sigma_2\wr\Sigma_d|}{|D_\lambda|}=\frac{2^d d!}{2^{a_1}\lambda_1^{a_1}a_1!\ldots 2^{a_k}\lambda_k^{a_1}a_k!}.$$

In particular, in the Hecke algebra $He(\Sigma_{2p},\Sigma_2\wr\Sigma_p)_{\mathbf{k}}$, where $\mathbf{k}$ is a field of characteristic $p>2$, thanks to the Waring and Fermat theorems, the formula massively simplifies to:

$$ind(e_\lambda)=\begin{cases} -1 & \text{for } \lambda=(p)\\ 0 & \text{otherwise.}\end{cases}$$

Let us finally relate these constructions to the setup of Section 1.1. We start with generalizing Proposition 1.4 to take into account the multiplicative structures. So, we consider $\mathbf{k}$-algebra M which is also an H-bimodule such that the multiplication factorizes to the H-biequivariant map $M\otimes_{\mathbf{k}[H]}M\longrightarrow M$.

Definition/Proposition 3.5. *Let M be as above. Then the $\mathbf{k}$-space ${}^H(M_H)$ has a natural structure of $\mathbf{k}$-algebra. We call this algebra the generalized Hecke algebra and denote by $GHe(M,H)$. If additionally there exists an H-biinvariant, multiplicative $\mathbf{k}$-basis X of M, then the multiplication in* GHe(M, H) *is given by the following formula. For $x,y\in H\backslash X/H$ we have:*

$$xy=\sum_{z\in H\backslash G/H}\mu_{xyz}z,$$

where the integer coefficients μ_{xyz} are computed in the following manner. ${}_{xyz}=0$ unless there exist representatives $x',y',z'\in X$ of x,y,z and $g\in H$

such that $x'gy' = z'$. In that case:

$$\mu_{xyz} = |(H \times H^{op})_{x'}|/|H_{x'}| \cdot |_{x'}H|,$$

where $(H \times G^{op})_{x'}, H_{x'}, {}_{x'}H$ are stabilizers of x' for respectively: the two-sided, right and left actions.

Proof: In order to show that the multiplication is well defined on GHe(M, H) we need to show two things: that the multiplication preserves invariants and that its result does not depend on the choice of representative in the coinvariants. In the computations to follow we denote the multiplication in M by a central dot, while the H-actions by lower dots. So, let $x, y \in M$ represent two elements of He(M, H) and $g \in H$. The crucial property of x (and y) following from the fact that they belong to the i invariants of coinvariants is that there exists $h \in H$ such that $g.x = x.h$ (and the analogous fact holds for y. Now using these and the invariance of the multiplication we obtain:

$$g.(x \cdot y) = (g.x) \cdot y = (x.h) \cdot y = x \cdot (h.y) = x \cdot (y.k) = (x \cdot y).k$$

which shows the first property. To see that the result of multiplication does not depend on representative x we compute:

$$(x.g) \cdot y = x \cdot (g.y) = x \cdot (y.h) = (x \cdot y).h$$

The independence on the representative y is shown similarly (even simpler). The formula for μ_{xyz} follows from the counting of the orbits for the left action on M_H. □

The most important instance of the above construction is that for a pair of finite groups $H \subset G$ and $M = \mathbf{k}[G]$. Then we have an observation which justifies our terminology and connects the Hecke algebras and the Yoneda algebras for plethysms.

Proposition 3.6. *Let $H \subset G$ be a pair of finite groups and $\mathbf{k}$ be any commutative ring. Then there is an isomorphism of $\mathbf{k}$-algebras:*

$$He(G, H)_{\mathbf{k}} \simeq GHe(\mathbf{k}[G], H).$$

Proof: Observe that $\mathrm{Hom}_{\mathbf{k}[G]-\mathrm{mod}}(\mathbf{k}[G], \mathbf{k}[G]) \simeq \mathbf{k}[G]$ as H-bimodules and $\mathbf{k}$-algebras. Thus we have:

$$He(G, H)_{\mathbf{k}} \simeq \mathrm{Hom}_{\mathbf{k}[G]-\mathrm{mod}}(\mathbf{k}[G/H], \mathbf{k}[G/H]) \simeq$$

$$^{H}(\mathrm{Hom}_{\mathbf{k}[G]-\mathrm{mod}}(\mathbf{k}[G], \mathbf{k}[G])_H) \simeq GHe(\mathbf{k}[G], H).$$

□

3.3. **On $\mathrm{Hom}_{\mathcal{P}}(S^p \circ S^2, S^p \circ S^2)$.**
In this subsection we apply the machinery of Hecke algebras to the description of multiplicative structure on endomorphisms of plethysms. Firstly, by Corollary 1.2 and Proposition 3.6 we have:

Proposition 3.7. *Let* $\mathbf{k}$ *be any field. For any* $d, i > 0$ *there is an isomorphism of* $\mathbf{k}$*-algebras:*

$$\mathrm{Hom}_{\mathcal{P}}(S^d \circ S^i, S^d \circ S^i) \simeq He(\Sigma_{di}, \Sigma_i \wr \Sigma_d)_{\mathbf{k}}.$$

In particular, the algebra $\mathrm{Hom}_{\mathcal{P}}(S^d \circ S^2, S^d \circ S^2)$ *is commutative.*

Proof: The first assertion follows from Corollary 1.6 and Proposition 3.6. The fact that the Hecke algebra $He(\Sigma_{2d}, \Sigma_2 \wr \Sigma_d)$ is commutative (in such case we say that $(\Sigma_{2d}, \Sigma_2 \wr \Sigma_d)$ is a Gelfand pair) is well known (see eg. [S, Ex. 2.3]. □

The main results of this section concern the case $i = 2$ and $d = p$, however some preliminary results hold in greater generality. Our main computation completes the description of the Yoneda algebra on $\mathrm{Ext}^{>0}_{\mathcal{P}_{2p}}(S^p \circ S^2, S^p \circ S^2)$ given in Section 3.1. Namely, we have:

Theorem 3.8. *For any* $x \in \mathrm{Hom}_{\mathcal{P}}(S^p \circ S^2, S^p \circ S^2)$ *and* $y \in \mathrm{Ext}^q_{\mathcal{P}}(S^p \circ S^2, S^p \circ S^2)$ *for* $q > 0$*, we have:*

$$xy = yx = \mathrm{ind}(x)y. \tag{3.2}$$

In particular, the algebra $\mathrm{Ext}^*_{\mathcal{P}}(S^p \circ S^2, S^p \circ S^2)$ *is commutative.*

Proof: In view of Theorem 3.1 and Proposition 3.7 we only need to establish the formula (3.2). We start with expressing ind in terms of endomorphisms of plethysms:

Proposition 3.9. *For any* $d > 2$*, restricting endomorphisms to the one-dimensional space yields the algebra homomorphism:*

$$He(\Sigma_{2d}, \Sigma_2 \wr \Sigma_d)_{\mathbf{k}} \simeq \mathrm{Hom}_{\mathcal{P}_{2d}}(S^d \circ S^2, S^d \circ S^2) \longrightarrow \mathrm{Hom}_{\mathbf{k}-\mathrm{mod}}(S^d \circ S^2(\mathbf{k}), S^d \circ S^2(\mathbf{k})) \simeq \mathbf{k}$$

which is ind.

Proof: We start with showing a similar but simpler fact:

Lemma 3.10. *For any* $d > 2$*, restricting endomorphisms to the one-dimensional space yields the algebra homomorphism:*

$$\mathbf{k}[\Sigma_{2d}] \overset{s}{\simeq} \mathrm{Hom}_{\mathcal{P}_{2d}}(I^{2d}, I^{2d}) \longrightarrow \mathrm{Hom}_{\mathbf{k}-\mathrm{mod}}(I^{2d}(\mathbf{k}), I^{2d}(\mathbf{k})) \simeq \mathbf{k}$$

which is $a_{\mathbf{k}}$.

Proof of the lemma: The isomorphism s is given by the Σ_d-action on the tensor power by permuting factors, but any permutation $\sigma \in \Sigma_{2d}$ when evaluated on the one-dimensional space is the identity map. This means that our algebra homomorphism sends $\sigma \in \Sigma_{2d}$ to $1 \in \mathbf{k}$, hence it is the augmentation. □

Proposition 3.9. follows from Lemma 3.10 and the commutative diagram:

$$\begin{array}{ccccc} He_{\mathbf{k}}(\Sigma_{2d}, \Sigma_2 \wr \Sigma_d]) & \simeq & \mathrm{Hom}_{\mathcal{P}_{2d}}(S^d \circ S^2, S^d \circ S^2) & \longrightarrow \\ \downarrow & & \downarrow & \\ \mathbf{k}[\frac{1}{2^d d!}][\Sigma_{2d}] & \simeq & \mathrm{Hom}_{\mathcal{P}_{2d}}(I^{2d}, I^{2d}) \otimes \mathbf{k}[\frac{1}{2^d d!}] & \longrightarrow \end{array}$$

$$\begin{array}{cccc} \longrightarrow & \mathrm{Hom}_{\mathbf{k}-\mathrm{mod}}(S^d\circ S^2(\mathbf{k}), S^d\circ S^2(\mathbf{k})) & \simeq & \mathbf{k} \\ & \downarrow & & \downarrow \\ \longrightarrow & \mathrm{Hom}_{\mathbf{k}-\mathrm{mod}}(I^{2d}(\mathbf{k}), I^{2d}(\mathbf{k}))\otimes \mathbf{k}[\frac{1}{2^d d!}] & \simeq & \mathbf{k}[\frac{1}{2^d d!}] \end{array}$$

□

Now we are ready for computing some Yoneda multiplications. In the sequel we will also denote by *ind* the one dimensional space $\mathbf{k}$ with the structure of $He(G,H)_{\mathbf{k}}$ induced by the algebra homomorphism $ind : He(G,H)_{\mathbf{k}} \longrightarrow \mathbf{k}$.

Lemma 3.11. $\mathrm{Hom}_{\mathcal{P}}(S^{2(1)}, S^p\circ S^2)$ *as a left* $\mathrm{Hom}_{\mathcal{P}}(S^p\circ S^2, S^p\circ S^2)$*-module is isomorphic to ind.*

Proof: $\mathrm{Hom}_{\mathcal{P}}(S^{2(1)}, S^p\circ S^2)$ is one-dimensional and is spanned by i given as the Frobenius twist $i' : I^{(1)} \longrightarrow S^p$ precomposed with S^2. Now we would like to compute the composite $e_\lambda \circ i$. Let us evaluate our functors on the one-dimensional space L spanned by a vector v. Let v_I and v_S stand for the corresponding vectors spanning respectively $S^{2(1)}(L)$ and $S^p\circ S^2(L)$. Then, obviously $i(v_I) = v_S$ and, by Proposition 3.9, $e_\lambda(v_S) = ind(e_\lambda)\cdot v_S$. Therefore:

$$e_\lambda \circ i(v_I) = ind(e_\lambda)\cdot v_S$$

which, since $\mathrm{Hom}_{\mathcal{P}}(S^{2(1)}, S^p\circ S^2)$ is one-dimensional, finishes the proof. □

Lemma 3.12. *For any* $j>0$ *such that* $\mathrm{Ext}^j_{\mathcal{P}}(S^p\circ S^2, S^p\circ S^2)$ *is non-zero, it is isomorphic to ind as a left* $\mathrm{Hom}_{\mathcal{P}}(S^p\circ S^2, S^p\circ S^2)$*-module.*

Proof: By Theorem 3.1, the module structure is the same for all j's. Hence it suffices to show our statement for $j=4$. Let us look at the spectral sequences converging to $\mathrm{HExt}^*_{\mathcal{P}}(R^p\circ S^2, S^p\circ S^2)$. Since the zeroth column survives in the both sequences, we have an isomorphism of left $\mathrm{Hom}_{\mathcal{P}}(S^p\circ S^2, S^p\circ S^2)$-modules:

$$\mathrm{Ext}^4_{\mathcal{P}}(S^p\circ S^2, S^p\circ S^2)\simeq \mathrm{Ext}^4_{\mathcal{P}}(S^{2(1)}, S^p\circ S^2).$$

Since $\mathrm{Ext}^*_{\mathcal{P}}(S^{2(1)}, S^p\circ S^2)$ is generated by $\mathrm{Hom}_{\mathcal{P}}(S^{2(1)}, S^p\circ S^2)$ as a right $\mathrm{Ext}^*_{\mathcal{P}}(S^{2(1)}, S^{2(1)})$-module, we have an isomorphism of left $\mathrm{Hom}_{\mathcal{P}}(S^p\circ S^2, S^p\circ S^2)$-modules:

$$\mathrm{Ext}^4_{\mathcal{P}}(S^{2(1)}, S^p\circ S^2). \simeq \mathrm{Hom}_{\mathcal{P}}(S^{2(1)}, S^p\circ S^2),$$

which finishes the proof. □

Now let us look at the right structures. We have an analogous fact:

Lemma 3.13. *For any* $j>0$ *such that* $\mathrm{Ext}^j_{\mathcal{P}}(S^p\circ S^2, S^p\circ S^2)$ *is non-zero, it is isomorphic to ind as a right* $\mathrm{Hom}_{\mathcal{P}}(S^p\circ S^2, S^p\circ S^2)$*-module.*

Proof: Analogously to Lemma 3.12 we reduce our task to showing that $\mathrm{Ext}^{2p-2}_{\mathcal{P}}(S^p\circ S^2, S^{2(1)})$ is isomorphic to *ind* as a right $\mathrm{Hom}_{\mathcal{P}}(S^p\circ S^2, S^p\circ S^2)$-module. Let $\alpha\in \mathrm{Ext}^{2p-2}_{\mathcal{P}}(S^p\circ S^2, S^{2(1)})$ be a non-zero element. Take $e_\lambda \in \mathrm{Hom}_{\mathcal{P}}(S^p\circ S^2, S^p\circ S^2)$, then $\alpha\circ e_\lambda = c\cdot\alpha$ and our task is to show that $c = ind(e_\lambda)$. We recall from the proof of Lemma 3.11 the element $i \in \mathrm{Hom}_{\mathcal{P}}(S^{2(1)}, S^p \circ S^2)$. By Lemma 3.11 we obtain:

$$c \cdot \alpha \circ i = \alpha \circ e_\lambda \circ i = \alpha \circ ind(e_\lambda) \cdot i,$$

which finishes the proof. □

Lemmas 3.12 and 3.13 give us formula (3.2). □

4. On the unit of the adjunction

In this section we look at the precomposition with S^2 as a functor. Our goal is to describe the unit of the adjunction which it generates. It may be used to obtain further Ext-computations for plethysms and also to categorify the precomposition with S^2 in the spirit of [C4]. We plan to develop these points in a further work.

We consider the functor

$$\mathbf{C}_{S^2} : \mathcal{P}_d \longrightarrow \mathcal{P}_{2d}$$

given as $\mathbf{C}_{S^2}(F) := F \circ S^2$. Obviously, by the Special Adjoint Functor theorem, $\mathbf{C}_{S^2}$ has the right adjoint functor:

$$\mathbf{K}_{S^2} : \mathcal{P}_{2d} \longrightarrow \mathcal{P}_d$$

which by the Yoneda lemma is explicitly given by:

$$\mathbf{K}_{S^2}(F)(V) = \mathrm{Hom}_{\mathcal{P}_{2d}}(\Gamma^{d,V} \circ S^2, F),$$

where $\Gamma^{d,V}(W) := \Gamma^d(W \otimes V^*)$ is a projective generator of $\mathcal{P}_d$. Since $\mathbf{C}_{S^2}$ is exact, it prolongs degreewise to the functor, which we shall also denote by $\mathbf{C}_{S^2}$, between the bounded derived categories. Then $\mathbf{K}_{S^2}$ as a right exact functor has the right derived functor $\mathbf{RK}_{S^2}$ given by:

$$\mathbf{RK}_{S^2}(F)(V) = \mathrm{RHom}_{\mathcal{P}_{2d}}(\Gamma^{d,V} \circ S^2, F).$$

and $\mathbf{C}_{S^2}$ and $\mathbf{RK}_{S^2}$ as functors between bounded derived categories are still adjoint.

Now it seems to be a general phenomenon concerning precomposition in $\mathcal{P}$ that the unit of the adjunction at the level of abelian categories ie. the composite $\mathbf{K}_F \circ \mathbf{C}_F$ and at the level of triangulated categories ie. the composite $\mathbf{RK}_F \circ \mathbf{C}_F$ admit quite explicit descriptions. The case of precomposition with $F = I^{(1)}$ was studied in [C3], [C4]. The situation there is quite interesting. In the abelian case $\mathbf{K}_{I^{(1)}} \circ \mathbf{C}_{I^{(1}}$ is isomorphic to the identity, since $\mathbf{C}_{I^{(1)}}$ is a full embedding. In the triangulated case it is not, but it is not far from it. Namely $\mathbf{RK}_{I^{(1)}} \circ \mathbf{C}_{I^{(1)}}$ is isomorphic to the functor of pretensoring with the graded space A (this space also appears in our Section 2.3 where it is denoted by A^*). This result had numerous applications (see [C3]) and also allowed categorification in [C4], which is potentially beneficial. Now, we shall observe a somewhat similar picture for the precomposition with S^2. Our main result, Theorem 4.10, explicitly describes $\mathbf{RK}_{S^2} \circ \mathbf{C}_{S^2}$ for $d = p$. However for $F = S^2$ already the case of $\mathbf{k}$ of characteristic zero, where there is no homological algebra at all is interesting. We give in Theorem.4.3 a full description of $\mathbf{K}_{S^2} \circ \mathbf{C}_{S^2}$ in that case. We recommend even the reader interested only in Theorem 4.10 to look also at Section 4.1, since the proof of Theorem 4.10, while much more intricate and subtle, does heavily use the ideas behind Theorem 4.3. Also comparing Theorems 4.3 and 4.10 explains to some extent the impression prevalent in all computations in Sections 2 and 3: that we have a large part of characteristic free structure reflected in the Hom-groups, and there is some homological addition emerging in the modular case, which at least for $d = p$ is controllable.

4.1. **Semisimple case.** In this subsection we assume that $d!$ is invertible in the ground field $\mathbf{k}$. Let

$$\mathbf{C}_{S^2} : \mathcal{P}_d \longrightarrow \mathcal{P}_{2d}$$

be the functor of precomposing with S^2 and $\mathbf{K}_{S^2}$ its right adjoint. We recall that explicitly:

$$\mathbf{K}_{S^2}(F)(V) = \mathrm{Hom}_{\mathcal{P}_{2d}}(\Gamma^{d,V} \circ S^2, F).$$

Our task is to describe the unit of this adjunction:

$$\mathbf{U} := \mathbf{K}_{S^2} \circ \mathbf{C}_{S^2}.$$

We would like to compute $\mathbf{U}(S^d_U)$, where $S^d_U(V) := S^d(V \otimes U)$ is an injective cogenerator of $\mathcal{P}_d$. To this end we start with computing $\mathbf{U}(I^d_U)$. We have:

$$\mathbf{U}(I^d_U) = \mathrm{Hom}_{\mathcal{P}_{2d}}(\Gamma^{d,V} \circ S^2, I^d_U \circ S^2) = \mathrm{Hom}_{\mathcal{P}_{2d}}(\Gamma^{d,V} \circ S^2, I^d \circ S^2) \otimes U^{\otimes d} =$$

$$= \bigoplus_{\gamma \in Polyh(d,2)} S^\gamma \otimes U^{\otimes d}, \tag{4.1}$$

where for $\gamma = \{\gamma_1^{a_1}, \dots \gamma_k^{a_k}\}$, S^γ stands for $S^{a_1} \otimes \ldots \otimes S^{a_k}$ (compare Section 1.2. The Σ_d-action on I^d_U induces an action on $\mathbf{U}(I^d_U)$, which is a diagonal action on the both factors in the tensor product

$$(\bigoplus_{\gamma \in Polyh(d,2)} S^\gamma) \otimes (U^{\otimes d}).$$

The action on $U^{\otimes d}$ is just by permuting factors in the tensor power. In order to describe the action on $\bigoplus_{\gamma \in Polyh(d,2)} S^\gamma$ it will be more convenient to replace $Polyh(d,2)$ with $Gr(d,2)$. We recall from Section 1.2 that we form our graph by connecting the numbers k and l whenever they occur in some γ_j. Then we can say that for the graph corresponding to γ we form S^γ by assigning to each single edge a copy of I, assigning to each double edge a copy of S^2 and tensoring them all. Then our Σ_d-action re-labels the copies of I and S^2 according to the Σ_d-action on $Gr(d,2)$ by re-numbering the vertices. In order to describe this action more explicitly we recall from Section 1.2 that the set of isomorphism classes of $Gr(d,2)$ is labeled by the set $\Lambda(d)$ of Young diagrams of weight d. Moreover, we have chosen in Section 3.2 for each $\lambda \in \Lambda(d)$ the graph G_λ belonging to the Σ_d-orbit labeled by λ. Our current situation is slightly different from that from Section 3.2, since we are interested in Σ_d-action instead of $\Sigma_2 \wr \Sigma_d$-action, hence we have undecorated graphs, but we can still use G_λ just forgetting about decoration. So, in our situation the isotropy group of G_λ is the ordinary dihedral group D_λ (or rather the semidirect product of dihedral groups). Then the action of D_λ on $S^{\gamma(G_\lambda)}$ is the following (we recall that the copies of V are indexed by the edges of the graph). Let

$$J_\lambda := \Sigma_2^{\times b_2} \rtimes \Sigma_{b_2}$$

where b_2 is the number of cycles of length 2 in G_λ ie. the multiplicity of 2 in λ. Then obviously $J_\lambda \lhd D_\lambda$ and the permutative action of D_λ on I^d factorize to the action of D_λ/J_λ on $S^{\gamma(G_\lambda)} = (I^d)_{J_\lambda}$. Then we pull back this to D_λ. Then by using the described above action of D_λ on $S^{\gamma(G_\lambda)}$ we obtain the isomorphism of $\mathbf{k}[\Sigma_d]$-modules:

$$\bigoplus_{\gamma\in Polyh(d,2)} S^\gamma \simeq \bigoplus_{\lambda\in\Lambda(d)} S^{\gamma(G_\lambda)} \otimes_{\mathbf{k}[D_\lambda]} \mathbf{k}[\Sigma_d].$$

Thus we can refine (4.1) to the form:

Proposition 4.1. *There is a natural in $U \in \mathrm{Vect}_{\mathbf{k}}$ isomorphism:*

$$\mathbf{U}(I_U^d) = \bigoplus_{\lambda\in\Lambda(d)} (S^{\gamma(G_\lambda)} \otimes_{\mathbf{k}[D_\lambda]} \mathbf{k}[\Sigma_d]) \otimes U^{\otimes d}.$$

In particular:

$$\mathbf{U}(I^d) = \bigoplus_{\lambda\in\Lambda(d)} S^{\gamma(G_\lambda)} \otimes_{\mathbf{k}[D_\lambda]} \mathbf{k}[\Sigma_d].$$

Therefore by applying Theorem 1.1 we obtain:

Proposition 4.2. *There is a natural in $U \in \mathrm{Vect}_{\mathbf{k}}$ isomorphism:*

$$\mathbf{U}(S_U^d) \simeq \bigoplus_{\lambda\in\Lambda(d)} V_\lambda(U),$$

where

$$V_\lambda(U) = S^{\gamma(G_\lambda)} \otimes_{\mathbf{k}[D_\lambda]} U^{\otimes d}.$$

In particular, if $2 \notin \lambda$ then the description of $V_\lambda(U)$ simplifies to:

$$V_\lambda(U) = (I_U^d)_{D_\lambda}.$$

Proof: Only the last formula requires some explanation. In that case we have $S^{\gamma(G_\lambda)} = I^d$, hence we get:

$$\mathbf{U}_\lambda(S_U^d) = I^d \otimes_{\mathbf{k}[D_\lambda]} U^{\otimes d} \simeq ((I \otimes U)^{\otimes d})_{D_\lambda} = (I_U^d)_{D_\lambda}.$$

□

Since S_U^d cogenerate $\mathcal{P}$, Proposition 4.2 determines $\mathbf{U}$. Namely we have:

Theorem 4.3. *There is an isomorphism:*

$$\mathbf{U} \simeq \bigoplus_{\lambda\in\Lambda(d)} \mathbf{U}_\lambda$$

and

$$\mathbf{U}_\lambda(F) = S^{\gamma(G_\lambda)} \otimes_{\mathbf{k}[D_\lambda]} S(F),$$

where $S(F) := \mathrm{Hom}_{\mathcal{P}}(I^d, F)$ *is "the Schur functor" in the sense of [M]. In particular, if* $2 \notin \lambda$ *then:*

$$\mathbf{U}_\lambda(F) = I^d \otimes_{\mathbf{k}[D_\lambda]} S(F).$$

Moreover:

$$\mathbf{U}_{(1^d)} \simeq id.$$

Proof of Theorem: By the Eilenberg-Watts theorem (or directly by the Yoneda lemma) $\mathbf{U}$ is representable, ie. there exists a bifunctor $(V, W) \mapsto X(V, W)$ contravariant in V such that:

$$\mathbf{U}(F)(V) = \mathrm{Hom}_{\mathcal{P}}(X(V, -), F),$$

(see eg. [C3, Section 2] for a detailed discussion of strict polynomial bifunctors and functors with parameters). Then on the one hand by Proposition 4.2 we have:

$$\mathbf{U}(S^d_U)(V) = \bigoplus_{\lambda\in\Lambda(d)} S^{\gamma(G_\lambda)} \otimes_{\mathbf{k}[D_\lambda]} S(F),$$

On the other hand by Yoneda lemma we get:

$$\mathrm{Hom}_{\mathcal{P}_d}(X(V, -), S^d_U) \simeq (X(V, U^*))^*.$$

By comparing these two expressions we obtain:

$$X(V, U) \simeq \bigoplus_{\lambda\in\Lambda(d)} (S^\lambda(V^*) \otimes U^{\otimes p})^{D_\lambda}$$

Hence, invoking again Theorem 1.1 we get:

$$\mathbf{U}_\lambda(F)(V) = \mathrm{Hom}_{\mathcal{P}}((S^\lambda(V) \otimes I^d)^{D_\lambda}, F) \simeq (\mathrm{Hom}_{\mathcal{P}}(S^{\gamma(G_\lambda)}(V^*) \otimes I^d, F))_{D_\lambda} \simeq$$
$$(S^{\gamma(G_\lambda)}(V) \otimes \mathrm{Hom}_{\mathcal{P}}(I^d, F))_{D_\lambda},$$

which proves our main assertion. The simplification for $2 \notin \lambda$ follows, like in Proposition 4.2 from the fact that $S^{\gamma(G_\lambda)} = I^d$ in that case. The further simplification for $\lambda = (1^d)$ is just the Yoneda lemma. □

4.2. **Case of $d = \mathrm{char}(\mathbf{k})$.**
Here we describe $\mathbf{RU}$ for $2 < d = p = \mathrm{char}(\mathbf{k})$. We still have

$$\mathbf{RU}(I^p_U) = \mathbf{U}(I^p_U) = \bigoplus_{\lambda\in\Lambda(p)} \mathbf{U}_\lambda(I^p_U) = \bigoplus_{\lambda\in\Lambda(p)} S^{\gamma(G_\lambda)} \otimes_{\mathbf{k}[D_\lambda]} \mathbf{k}[\Sigma_p] \otimes U^{\otimes p},$$

since it follows directly from the Yoneda lemma. We shall compute $\mathbf{RU}(S^p_U)$. Let us first observe that $S^p_U \circ S^2$ is a direct summand in $S^p_U \circ I^2$ and the latter by the Cauchy formula and the Littlewood-Richardson rule has a filtration with quotients being Schur functors. Analogously $\Gamma^p_U \circ S^2$ has a filtration with the quotients being Weyl functors. Therefore

$$\mathrm{Ext}^j_{\mathcal{P}}(\Gamma^{p,V} \circ S^2, S^p_U \circ S^2) = 0 \text{ for } j > 0, \tag{4.2}$$

which means that

$$\mathbf{RU}(S^p_U) \simeq \mathbf{U}(S^p_U).$$

We shall compute $\mathbf{U}(S^p_U)$ by comparing it with $\mathbf{U}(I^p_U)$ by means of the maps:

$$m_* : \mathbf{U}(I^p_U)_{\Sigma_p} \longrightarrow \mathbf{U}(S^p_U),$$

$$c_* : \mathbf{U}(S^p_U) \longrightarrow \mathbf{U}(I^p_U)^{\Sigma_p},$$

induced respectively by the multiplication and comultiplication map in the symmetric algebra. Their composite is induced by the symmetrization map $I^p \longrightarrow I^p$, thus it can be interpreted as the norm map N for the Σ_p-module $\mathbf{U}(I^d_U)$. Therefore we have:

Proposition 4.4. *The composition $c_* \circ m_*$ restricted to $(\mathbf{U}_\lambda(I^p_U))_{\Sigma_p}$ is an isomorphism for $\lambda \in \Lambda(p) \setminus \{(1^p), (p)\}$. In the special cases we have:*

(1) *For $\lambda = (1^p)$ the norm map*

$$N : (\mathbf{U}_{(1)^p}(I^p_U))_{\Sigma_p} \simeq) S^p_U \longrightarrow (\mathbf{U}_{(1^p)}(I^p_U))^{\Sigma_p} \simeq \Gamma^p_U$$

has the kernel and cokernel isomorphic to $I^{(1)}_U$.

(2) *$\lambda = (p)$ the norm map*

$$N : (\mathbf{U}_{(p)}(S^p_U))_{D_p} \simeq (I^p_U)_{D_p} \longrightarrow (\mathbf{U}_{(p)}(S^p_U))^{D_p} \simeq (I^p_U)^{D_p}$$

has the kernel and cokernel isomorphic to $I^{(1)}_U$.

Proof: The part about N being an isomorphisms follows from the fact that in this case the group D_λ has rank prime to p. The description of the kernel and cokernel of norm in the case of $\lambda = (1^p)$ is an elementary exercise. Let us now look at the case $\lambda = (p)$. Here we have:

Lemma 4.5. *There are compatible with N decompositions:*

$$(I^p_U)_{D_p} \simeq S^p_U \oplus X, \qquad (I^p_U)^{D_p} \simeq \Gamma^p_U \oplus X',$$

and the norm $N : X \longrightarrow X'$ is an isomorphism.

Proof of the Lemma: It follows from general properties of norms. Namely, the inclusion $\Gamma^p_U = (I^p_U)^{\Sigma_p} \subset (I^p_U)^{D_p}$ is just the inclusion of invariants of a group into the invariants of subgroup. Since $p \nmid [\Sigma_p : D_p]$, this inclusion splits by another incarnation of the norm sometimes called the transfer: $m \mapsto \sum_{g \in \Sigma_p/D_p} gm$. We remark that all these constructions do not require the normality of subgroup (Σ_p/D_p stands for some set of representatives of the cosets etc.). We obtain the analogous decomposition for invariants, and since all the maps are expressed in terms of Σ_p-action, they commute with N. The fact that the norm restricted to X is an isomorphism again follows from the fact that $p \nmid [\Sigma_p : D_p]$. □

Our assertion follows immediately from Lemma 4.5 and the part about $\lambda = (1^p)$. □

Then we analyze the kernel and cokernel of m_* and c_*. We start with collecting some auxiliary higher-Ext computations:

Lemma 4.6. *We have natural in $V, U \in \mathrm{Vect}_{\mathbf{k}}$ isomorphisms:*

(1)

$$\mathrm{Ext}^j_{\mathcal{P}_{2p}}(\Gamma^{p,V} \circ S^2, I_U^{(1)} \circ S^2) = \begin{cases} V^{(1)} \otimes U^{(1)} & \text{for } j \leq 2p-2,\ j = 4k \\ 0 & \text{otherwise.} \end{cases}$$

(2)

$$\mathrm{Ext}^{p-1}_{\mathcal{P}_{2p}}(\Gamma^{p,V} \circ S^2, \Lambda^p_U \circ S^2) = \mathrm{Ext}^{p-2}_{\mathcal{P}_{2p}}(\Gamma^{p,V} \circ S^2, \Lambda^p_U \circ S^2) = V^{(1)} \otimes U^{(1)}.$$

Proof: We first observe that by Theorem 2.5 for $F = S^2$ (or rather the explicit computation of the dimensions of Ext-groups given in the proof of Theorem 2.6) we get:

$$\mathrm{Ext}^j_{\mathcal{P}_{2p}}(\Gamma^p \circ S^2, S^{2(1)}) \simeq \mathrm{Ext}^j_{\mathcal{P}_{2p}}(S^{2(1)}, S^p \circ S^2) \simeq$$

$$\begin{cases} \mathbf{k} & \text{for } j \leq 2p-2,\ j = 4k \\ 0 & \text{otherwise,} \end{cases}$$

which agrees numerically with the formula (1) for $V = U = \mathbf{k}$. Now, since the both sides of (1) are additive with respect to V and U, the formula holds numerically for all V, U. Then observe that $\mathrm{Ext}^j_{\mathcal{P}_{2p}}(\Gamma^{p,V} \circ S^2, I_U^{(1)} \circ S^2)$ as functor in V, U is a strict polynomial bifunctor of degree p with respect to the both variables. Thus it is of the form $\bigoplus V^{(1)} \otimes U^{(1)}$, by [T2]. This shows that (1) also holds functorially.

In order to get (2) we take the de Rham complex with parameter $R_{p,U}$ and consider the spectral sequences converging to $\mathrm{HExt}^*_{\mathcal{P}}(\Gamma^{p,V} \circ S^2, R_{p,U} \circ S^2)$. By the Cartier theorem and (1) the second spectral sequence degenerates at E^2 and we get:

$$\mathrm{HExt}^j_{\mathcal{P}}(\Gamma^{p,V} \circ S^2, R_{p,U} \circ S^2) = \begin{cases} V^{(1)} \otimes U^{(1)} & \text{for } j \leq 2p-1,\ j = 4k \ \text{ or } \ j = 4k+1 \\ 0 & \text{otherwise,} \end{cases}$$

Now, we look at the first page of the first spectral sequence. Since $(S^i_U \otimes \Lambda^{p-i}_U) \circ S^2$ are injective for $0 < i < p$ and by $(4,2)$, we can have nontrivial higher Exts in the pth column, which is $\mathrm{Ext}^*_{\mathcal{P}_{2p}}(\Gamma^{p,V} \circ S^2, \Lambda^p_U \circ S^2)$. Since by the dimension argument these higher Exts survive, we get:

$$\mathrm{Ext}^{p-1}_{\mathcal{P}_{2p}}(\Gamma^{p,V} \circ S^2, \Lambda^p_U \circ S^2) = \mathrm{HExt}^{2p-1}_{\mathcal{P}}(\Gamma^{p,V}, R_{p,U} \circ S^2) = V^{(1)} \otimes U^{(1)},$$

and

$$\mathrm{Ext}^{p-2}_{\mathcal{P}_{2p}}(\Gamma^{p,V} \circ S^2, \Lambda^p_U \circ S^2) = \mathrm{HExt}^{2p-2}_{\mathcal{P}}(\Gamma^{p,V}, R_{p,U} \circ S^2) = V^{(1)} \otimes U^{(1)},$$

which finishes the proof. □

Then we have:

Proposition 4.7. *Both m_* and c_* have kernel and cokernel isomorphic to $I_U^{(1)}$. The kernel of m_* is located in $(\mathbf{U}_{(p)}(I^p_U))_{\Sigma_p} = (I^p_U)_{D_p}$, the cokernel of c_* is located in $(\mathbf{U}_{(1^p)}(I^p_U))^{\Sigma_p} = \Gamma^p_U$,*

Proof: We start with investigating m_*. Let us recall the bar complex with parameter $B_{p,U}$ (see eg. [Tot]). It is an exact sequence, which extends the projection $m' : I^p \longrightarrow S^p$:

$$0 \longrightarrow \Lambda^p_U \longrightarrow \ldots \longrightarrow \bigoplus_{[2]\subset[p]} \Lambda^2_U \otimes I^{p-2}_U \xrightarrow{d_1} I^p_U \xrightarrow{m'} S^p_U \longrightarrow 0.$$

Let us describe explicitly the differential d_1. Namely it sends $x_1 \wedge x_2 \otimes y_1 \otimes \ldots y_{p-2}$ from the copy of $\Lambda^2 \otimes I^{p-2}$ labeled by the inclusion $[2] \subset [p]$ onto $\{i, j\}$ to:

$$y_1 \otimes \ldots y_{i-1} \otimes x_1 \otimes \ldots y_{j-2} \otimes x_2 \otimes \ldots - y_1 \otimes \ldots y_{i-1} \otimes x_2 \otimes \ldots y_{j-2} \otimes x_1 \otimes \ldots,$$

ie. we insert x_1, x_2 at places i, j and antysymmetrize them. Thus one sees that $\mathrm{coker}(d_1) = (I^p_U)_{\Sigma_d}$, which will be important later.

Now, we consider the spectral sequence E converging to $\mathrm{HExt}_{\mathcal{P}_{2p}}(\Gamma^{p,V} \circ S^2, B_{p,U} \circ S^2)$. Observe that $E^{0p}_2 = \mathrm{coker}(m_*)$ and $E^{0,p-1}_2 = \ker(m_*)$. Since, like in the proof of Lemma 4.6.(2) ,we have higher Exts only in the last column, which is $\mathrm{Ext}^*_{\mathcal{P}_{2p}}(\Gamma^{p,V} \circ S^2, \Lambda^p_U \circ S^2)$, our assertion follows from Lemma 4.6.(2).

The case of c_* is slightly more involved computationally. This time we start with $B'_{p,U}$, a parametrized version of another variant of bar complex from [Tot]:

$$0 \longrightarrow \Gamma^p_U \longrightarrow \ldots \longrightarrow \bigoplus_{[2]\subset[p]} \Gamma^2_U \otimes I^{p-2}_U \longrightarrow I^p_U \longrightarrow \Lambda^p_U \longrightarrow 0.$$

Then we apply to $B'_{p,U}$ the functor $\mathrm{Hom}_{\mathcal{P}}(\Lambda^{p,(-)}, -)$. In order to analyze the resulting complex we observe that $B'_{p,U}$ may be interpreted as augmented resolution of Γ^p_U admissible for $\mathrm{Hom}_{\mathcal{P}}(\Lambda^{p,(-)}, -)$. Luckily, the relevant Ext-groups: $\mathrm{Ext}^*_{\mathcal{P}}(\Lambda^{p,V}, \Gamma^p_U)$ are known. Namely, they were computed in [A] for $U = V = \mathbf{k}$, in [C2] for $U = \mathbf{k}$ and an arbitrary V and it is an easy exercise to extend the computation to the case of an arbitrary U (also the elementary methods of [J] can be applied here). The result is:

$$\mathrm{Ext}^j_{\mathcal{P}}(\Lambda^{p,V}, \Gamma^p_U) = \begin{cases} V^{(1)} \otimes U^{(1)} & \text{for } j = p-1,\ j = p-2, \\ \Gamma^p(V \otimes U) & \text{for } j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

This allows one to describe explicitly the complex $\mathrm{Hom}_{\mathcal{P}}(\Lambda^{p,(-)}, B'_{p,U})$ and its cohomology. Namely, also taking into account the known from [FFSS] descriptions of Hom-maps between exponential functors we can rewrite $\mathrm{Hom}_{\mathcal{P}}(\Lambda^{p,(-)}, B'_{p,U})$ as:

$$0 \longrightarrow \Lambda^p_U \longrightarrow \ldots \longrightarrow \bigoplus_{[2]\subset[p]} \Lambda^2_U \otimes I^{p-2}_U \longrightarrow I^p_U \longrightarrow \Gamma^p_U \longrightarrow 0.$$

Moreover, we have:

$$H^j(\mathrm{Hom}_{\mathcal{P}}(\Lambda^{p,(-)}, B'_{p,U})) = \begin{cases} I_U^{(1)} & \text{for } j=0,\ j=1 \\ 0 & \text{otherwise.} \end{cases}$$

Finally we apply to our complex the Kuhn duality and obtain the complex $C_{p,U}$:

$$0 \longrightarrow S_U^p \xrightarrow{c'} I_U^p \xrightarrow{m_2} \bigoplus_{[2]\subset[p]} \Lambda^2 \otimes I^{p-2} \longrightarrow \ldots \longrightarrow \Lambda_U^p \longrightarrow 0.$$

where $c' : S_U^p \longrightarrow I_U^p$ is the comultiplication map. Also this time we have: $\ker(m_2) = (I_U^p)^{\Sigma_p}$. Its cohomology is still:

$$H^j(C_{p,U}) = \begin{cases} I_U^{(1)} & \text{for } j=0,\ j=1, \\ 0 & \text{otherwise.} \end{cases} \tag{4.3}$$

Then we consider the first spectral sequence E converging to $\mathrm{HExt}^*_{\mathcal{P}_{2p}}(\Gamma^{p,V} \circ S^2, C_{p,U} \circ S^2)$. We observe that $E_2^{00} = \ker(c_*)$ and $E_2^{0,1} = \mathrm{coker}(c_*)$ and they survive. Hence in order to identify these groups we need to look at the second spectral sequence F. By (4.3) F_2^{**} consists of the first two columns where we have $\mathrm{Ext}^j_{\mathcal{P}_{2p}}(\Gamma^{p,(-)} \circ S^2, I_U^{(1)} \circ S^2)$. These groups were computed in Lemma 4.6.(1). Since they are concentrated in even degrees, they survive, hence we get:

$$\ker(c_*) = E_2^{00} = \mathrm{HExt}^0_{\mathcal{P}_{2p}}(\Gamma^{p,V} \circ S^2, C_{p,U} \circ S^2) = F_2^{00} = I^{(1)},$$

and

$$\mathrm{coker}(c_*) = E_2^{01} = \mathrm{HExt}^1_{\mathcal{P}_{2p}}(\Gamma^{p,V} \circ S^2, C_{p,U} \circ S^2) = F_2{}^0 1 = I^{(1)},$$

which finishes the proof. □

Now can conclude:

Proposition 4.8. *There is a natural in $U \in \mathrm{Vect}_{\mathbf{k}}$ decomposition:*

$$\mathbf{U}(S_U^p) = \bigoplus_{\lambda \in \Lambda(p)} V_\lambda(U),$$

and the two overlapping isomorphisms:

$$V_\lambda(U) = m_*(\mathbf{U}_\lambda(I_U^p)_{\Sigma_p} = (S^\lambda \otimes U^{\otimes p})_{D_\lambda}, \quad \text{for } \lambda \neq (p)$$

and

$$V_\lambda(U) \simeq c_*(V_\lambda) = \mathbf{U}_\lambda(I_U^p)^{\Sigma_p} = (S^\lambda \otimes U^{\otimes p})^{D_\lambda}, \quad \text{for } \lambda \neq (1^p).$$

Proof: Thanks to Prop. 4.7 we know that m_* produces an embedding

$$\bigoplus_{\lambda \in \Lambda(p)\setminus\{(p)\}} (S^\lambda \otimes U^{\otimes p})_{D_\lambda} \subset \mathbf{U}(S_U^p).$$

We need to find additionally S_U^p and $(I_U^d)^{D_p}$ inside $\mathbf{U}(S_U^p)$. We start with S_U^p.

Lemma 4.9. *The unit map:*

$$S^p_U \longrightarrow \mathbf{U}(S^p_U)$$

is an embedding which splits naturally in U and its image is contained in $m_*(\mathbf{U}_{(1^p)}(I^d_U)_{\Sigma_d})$.

Proof: The unit map $S^p_U \longrightarrow \mathbf{U}(S^p_U)$ can be interpreted as precomposing on Hom-maps:

$$S^P_U \simeq \mathrm{Hom}_{\mathcal{P}}(\Gamma^{p,(-)}, S^p_U) \subset \mathrm{Hom}_{\mathcal{P}}(\Gamma^{p,(-)} \circ S^2, S^p_U \circ S^2) = \mathbf{U}(S^p_U).$$

Therefore, by Theorem 2.1 it naturally splits. The second statement follows from the fact that the image of of the unit map:

$$I^p_U \longrightarrow \mathbf{U}(I^p_U)$$

is exactly $\mathbf{U}_{(1^p)}(I^p_U)$, which immediately follows from the Exponential Formula. □

Thus we can decompose $\mathbf{U}(S^p_U)$ as $X \oplus S^p_U$ with

$$X \supset \bigoplus_{\lambda \in \Lambda(p) \backslash \{(p)\}} (S^\lambda \otimes U^{\otimes p})_{D_\lambda}.$$

Then let $\pi : \mathbf{U}(S^p_U) \longrightarrow \mathbf{U}(S^p_U)$ be the projection onto X with the kernel S^p_U. Then we find our copy of $(I^p_U)^{D_p}$ by taking $\pi(c_*^{-1}((I^p_U)^{D_p}))$. □

Then, finally, we obtain the following description of the derived functor of $\mathbf{U}$:

Theorem 4.10. *We have:* $\mathbf{RU} = \bigoplus_{\lambda \in \Lambda(p)} \mathbf{RU}_\lambda$ *and:*

$$\mathbf{RU}_\lambda = \mathbf{U}_\lambda(F) = S^{\gamma(G_\lambda)} \otimes_{\mathbf{k}[D_\lambda]} S(F), \quad \text{for} \quad \lambda \notin \{(p), (1^p)\}.$$

In particular, if $2 \notin \lambda \notin \{(p), (1^p)\}$:

$$\mathbf{U}_\lambda(F) = I^d \otimes_{\mathbf{k}[D_\lambda]} S(F).$$

Moreover:

$$\mathbf{RU}_{(1^d)} \simeq id,$$

and

$$\mathbf{RU}_{(p)}(F) = \mathrm{RHom}_{\mathcal{P}}((I^{p,(-)})^{D^p}, F).$$

Proof: We proceed along the lines of the proof of Theorem 4.3, with the Eilenberg-Watts theorem replaced by the Neeman Representability theorem [N]. The formula for $\lambda \notin \{(p), (1^p)\}$ is exactly the same as in Theorem 4.3 because $p \nmid |D_\lambda|$ in this case and $(I^{p,V})^{D_\lambda}$ is projective. The formula for $\mathbf{U}_{(1^p)}$ is just the Yoneda lemma. □

References

[A] K. Akin , *Extensions of symmetric tensors by alternating tensors*, Journal of Algebra **121** (1989) no. 2, 358-363.

[ABW] K. Akin, D.A. Buchsbaum, J. Weyman, *Schur functors and Schur complexes*, Advances in Math. **44** (1982), no. 3, 207–278.

[C1] M. Chałupnik, *Extensions of strict polynomial functors*, Ann. Sci. Éc. Norm. Sup., 4 série **38** (2005), 773–792.

[C2] M. Chałupnik, *Koszul duality and extensions of exponential functors*, Advances in Math. **218** (2008), 969–982.

[C3] M. Chałupnik, *Derived Kan extension for strict polynomial functors*, Int. Math. Research Notices **2015** (2015), no. 20, 10017–10040.

[C4] M. Chałupnik, *Affine strict polynomial functors and formality*, Advances in Mathematics **320** (2017), 652–673.

[CR] C.W. Curtis, I. Reiner, *Methods of representation theory with applications to finite groups and orders, vol. 1*, John Wiley & Sons (1981).

[F] H. Foulkes, *Concomitants of the cquintic and sextic up to degree four in the coefficients of the ground form*, J. London Math. Soc. **25** (1950), 205-209.

[FFSS] V. Franjou, E. Friedlander, A. Scorichenko, A. Suslin, *General Linear and Functor Cohomology over Finite Fields*, Annals of Math. **150** (1999), no. 2, 663–728.

[FLS] V. Franjou, J. Lannes, L. Schwartz. *Autour de la cohomologie de MacLane des corps finis*, Invent. Math. **115** (1994), 513-538.

[FP] V. Franjou, T. Pirashvili, *Strict polynomial functors and coherent functors*, Manuscripta Mathematica **127** (2008), 23-53.

[G] J.A. Green, *Polynomial representations of GL_n*, Lecture Notes in Math. **830**, Springer, Berlin 2007.

[J] P. Jaśniewski, *On homological properties of strict polynomial functors of degree p* Journal of Algebra **618** (2023), 141-164.

[K1] N. Kuhn, *Generic representations of the finite general linear groups and the Steenrod algebra: II*, K-theory **8** (1994), 395-428.

[K2] N. Kuhn, *Generic representations of the finite general linear groups and the Steenrod algebra: III*, K-theory **9** (1995), 273-303.

[M] S. Martin, *Schur algebras and representation theory*, Cambridge Tracts in Mathematics, 112. Cambridge University Press, Cambridge, 2008.

[N] A. Neeman, *The Grothendieck duality theorem via Bousfield's techniques and Brown representability*, Journal of the American Mathematical Society **9** (1) (1996), 205-236.

[S] J. Saxl, *On multiplicity-free permutation representations*, Finite geometries and designs, (Proc. Conf., Chelwood Gate, 1980), London Math. Soc. Lecture Note Ser., vol. 49, Cambridge University Press (1981), pp. 337-353.

[T1] , A. Touzé, *Troesch complexes and extensions of strict polynomial functors*, Ann. Sci. Éc. Norm. Sup., **45** (2012) no. 1, 53-99.

[T2] A. Touzé, *A functorial control of integral torsion in homology*, Fund. Math. **237** (2017), 135-163.

[Tot] B. Totaro, *Projective resolutions of representations of $GL(n)$*, Journal für die reine und angewandte Mathematik **482** (1997), 1-14.

[Tr] A. Troesch, *Une formule pour les extensions de foncteurs composés*, Fund. Math. **177** (2003), 55-82.

Instytut Matematyki, Uniwersytet Warszawski, Warszawa, Poland
Email address: betley@mimuw.edu.pl

Instytut Matematyki, Uniwersytet Warszawski, Warszawa, Poland
Email address: mchal@mimuw.edu.pl