Some problems 2015

June 3, 2015

♦ - leftovers from the lecture

1 Exercises for the first lecture

1.1 Let G be a topological group. Show that if a subgroup $H \subset G$ is open, then it is closed. Show that component of 1 is a subgroup.

1.2 Show that $\pi_1(G)$ is abelian.

1.3 Let G be a connected topological group. Let $p : \widetilde{G} \to G$ be a connected covering. Choose an inverse image of 1. Show that $\widetilde{G}$ has a natural group structure, such that p is a homomorphism.

1.4 Show that any discrete normal subgroup of a connected topological group G is in the center $Z(G)$.

1.5 Suppose G is a connected topological group, $Z(G)$ is discrete. Then $G/Z(G)$ has trivial center.

1.6 Let $\mathbb{H}$ denote quaternions. Show that the set of the continuous maps of rings $map(\mathbb{C}, \mathbb{H})$ is parameterized by the 2-dimensional sphere.

1.7 Prove that $Aut(\mathbb{H})$, the group of continuous automorphisms of quaternions, is isomorphic to $SO(3)$.

1.8 What is the dimension of $U(n)$ and $SU(n)$?

1.9 Let ϕ be a nondegenerate form in $\mathbb{R}^n$ of the type (k, ℓ), where $k + \ell = n$. Let $O(k, \ell) \subset GL_n(\mathbb{R})$ denote the group of linear transformations preserving ϕ. How many topological components has this group?

1.10 We identifying the quaternions with $\mathbb{C}^2 \simeq \mathbb{C} \oplus j\mathbb{C} = \mathbb{H}$ we have embedding $GL_n(\mathbb{H}) \subset GL_{2n}(\mathbb{C})$. Define the compact symplectic group as $Sp(n) = U(2n) \cap GL_n(\mathbb{H})$. Prove $Sp(n) \subset SU(2n)$. What is the dimension of $Sp(n)$?

1.11 Let ω be a standard symplectic form in $\mathbb{C}^{2n}$, and let $Sp_n(\mathbb{C})S$ (a.k.a. $Sp(n, \mathbb{C})$) denote the subgroup of $GL_{2n}(\mathbb{C})$ preserving the form ω. Show that $Sp(n) \simeq Sp_n(\mathbb{C}) \cap U(2n)$ (be careful with the order of coordinates). What is the dimension of $Sp_n(\mathbb{C})$?

1.12 Let $SO_n(\mathbb{C})$ (a.k.a. $SO(n, \mathbb{C})$) denote the subgroup of $GL_n(C)$ preserving the standard nondegenerate quadratic form. Construct nontrivial maps
a) $SL_2(\mathbb{C}) \to SO_3(\mathbb{C})$
b) $SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \to SO_4(\mathbb{C})$
c) $SL_4(\mathbb{C}) \to SO_6(\mathbb{C})$
d) $Sp_2(\mathbb{C}) \to SO_5(\mathbb{C})$

1.13 Let $PGL_n(\mathbb{C}) = GL_n(\mathbb{C})/Z(GL_n(\mathbb{C}))$. Find an embedding $PGL_n(\mathbb{C}) \hookrightarrow GL_m(\mathbb{C})$ for some m.

1.14 ♦ $U(n)$, $SU(N)$, $SO(n)$, $Sp(n)$ are connected, $O(n)$ has two components

1.15 ♦ $\pi_1(U(n)) = \mathbb{Z}$, $\pi_1(SU(n)) = 1$, $\pi_1(SO(n)) = \mathbb{Z}_2$ for $n \geq 3$ (long exact sequence of homotopy groups needed)

1.16 ♦ Check that every map $\mathbb{H}^n \to \mathbb{H}^n$ which is a map of left $\mathbb{H}$-modules is determined by a matrix $M_{n\times n}(\mathbb{H})$

$$(v_1, v_2, \ldots, v_n) \mapsto (v_1, v_2, \ldots, v_n) \cdot A\,,$$

where $\cdot$ denotes the matrix multiplication. Any map of right $\mathbb{H}$-modules is of the form

$$(v_1, v_2, \ldots, v_n)^T \mapsto A \cdot (v_1, v_2, \ldots, v_n)^T\,.$$

1.17 ♦ Let's define the group $Sp(n)$ as the automorphism of the left $\mathbb{H}$-module $\mathbb{H}^n$ preserving the norm. Show that the elements of $Sp(n)$ preserve the form $\mathbb{H}^n \times \mathbb{H}^n \to \mathbb{H}$ given by $(v, w) = \sum_{i=1}^{n} v_i \overline{w}_i$.

1.18 ♦ The real symplectic group $Sp_n(\mathbb{R}) \subset GL_{2n}(\mathbb{R})$ (appears in real symplectic geometry or in classical mechanics) is noncompact. Prove the Iwasawa decomposition for that group with $K = U(n)$.

2 Exercises for the second lecture

2.1 Show that $\mathbb{R}^3$ with the vector product $\times$ is a Lie algebra isomorphic to $so(3)$.

2.2 Compare the Lie algebra of upper-triangular 3×3 matrices with 0's on the diagonal with the Lie algebra generated by x and $\frac{d}{dx}$ acting on the polynomial ring $\mathbb{C}[x]$.

2.3 Let A be an algebra (not necessarily associative). The derivations of A are defined as

$$Der(A) = \{\phi \in Hom_{Vect\ spaces}(A, A) \,|\, \forall a, b \in A\ \phi(ab) = \phi(a)b + a\phi(b)\}.$$

Check that the commutator of two derivations is a derivation.

2.4 Compute few terms of Baker-Campbell-Hausdorff formula. (At least the third term.)

2.5 ♦ Show that for $X, Y \in M_{n\times n}(\mathbb{C})$ the $exp(tX)exp(tY)) = exp(\sum_{n=0}^{\infty} t^n A_n)$, where A_n is a Lie polynomial in X, Y, ie. can be expressed by X, Y, $+$, $-$, $[-,-]$ and scalar multiplication.

2.6 Show that exp for $SU(2)$ is surjective. At which points is it a submersion?

2.7 For which groups: $GL_n^+(\mathbb{R})$, $SL_n(\mathbb{R})$, $GL_n(\mathbb{C})$, $SL_n(\mathbb{C})$, B_n (upper triangular), N_n (upper triangular with 1's on the diagonal) exp is surjective?

2.8 Show that if G is connected, $[X, Y] = 0$ for any $X, Y \in \mathfrak{g}$, then G is abelian.

3 Exercises for the third lecture

3.1 Describe all real and complex Lie algebras of the dimension ≤ 3. Which real algebras become isomorphic after tensoring with $\mathbb{C}$?

3.2 Let $f : G \to H$ be a map of connected Lie groups. Show that if $Df : \mathfrak{g} \to \mathfrak{h}$ is an isomorphism, then f is surjective with discrete kernel contained in $Z(G)$.

3.3 What is the center of $SO(n)$?

3.4 Check the formula

$$\frac{d}{dt} e^{A+tB} = e^A \left(B - \frac{[A,B]}{2!} + \frac{[A,[A,B]]}{3!} - \frac{[A,[A,[A,B]]]}{4!} + \ldots \right).$$

3.5 ♦ Let $X \subset M_{n\times n}(\mathbb{R})$ be the space of symmetric positive definte matrices. Show that X is diffeomorphic to an affine space.

3.6 ♦ Let $X \subset M_{n\times n}(\mathbb{C})$ be the space of hermitian positive definte matrices. Show that X is diffeomorphic to a real affine space.

3.7 Does there exist a compact Lie group with the Lie algebra isomorphic to $\mathfrak{sl}_2(\mathbb{R})$?

3.8 The disc $\{z \in \mathbb{C} \ : \ |z| < 1\}$ does not admit a structure of a complex Lie group.

3.9 Compute the differential of the map $GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$, $A \mapsto A^2$ in the direction X. Show that it does not vanish if A and X are symmetric, A positive definite.

3.10 Does there exist a quaternionic determinant, i.e. a map $\phi : M_{n\times n}(\mathbb{H}) \to \mathbb{H}$ or $\phi : GL_n(\mathbb{H}) \to \mathbb{H}^*$ satisfying $\phi(AB) = \phi(A)\phi(B)$?
- what if we additionally assume that $\phi(diag(a, 1, 1, \ldots, 1)) = a$ for $a \in \mathbb{H}$?
- what if we additionally assume that ϕ is surjective ?
- what if we additionally assume that $\phi(A) = \det(A)$ for $A \in GL_n(\mathbb{C})$?

3.11 Describe the set $Tr(exp(\mathfrak{sl}_2(\mathbb{R}))) \subset \mathbb{R}$ and compare with $Tr(SL_2(\mathbb{R}))$.

3.12 Knowing the center of $SU(n)$ and $\pi_1(SU(n))$ use Lie theorem to say how many Lie groups have the Lie algebra $\mathfrak{su}(n)$. (Of course count up to an isomorphism.)

3.13 Knowing the center of $SO(n)$ and $\pi_1(SO(n))$ use Lie theorem to list all the Lie groups with the Lie algebra $\mathfrak{so}(n)$. Count up to an isomorphism preserving the Lie algebra. The answer depends on the parity of n.

3.14 Compute the center of $Sp(n)$ and $\pi_1(Sp(n))$ (hint $Sp(n)/Sp(n-1) = S^{4n-1}$). Using Lie theorem list all the Lie groups with the Lie algebra $\mathfrak{sp}(n)$.

3.15 Let Q be the set of solutions of one quadratic equation in $\mathbb{P}^n(\mathbb{R})$ or in $\mathbb{P}^n(\mathbb{C})$. Assume the Q is smooth and nonempty. Show that Q is a homogenous space, i.e. there exist a Lie group G acting on Q in a transitive way. Show that the group G can be chosen to be compact. Find the stabilizer of a point.

3.16 Lagrangian Grassmanian $LG_n(\mathbb{K})$ is the set of isotropic subspaces of maximal dimension (isotropic with respect to the symplectic form) in $\mathbb{K}^{2n}$ for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Show that $LG_n(\mathbb{K})$ is a homogenous space for $Sp_n(\mathbb{K})$. Show that $LG_n(\mathbb{R}) = U(n)/O(n)$. Show that $LG(n)$ is homogeneous with respect to $Sp(n)$. Find a stabilizer of a point.

3.17 Show that the map $\phi : U(n)/O(n) \twoheadrightarrow S^1$, $\phi([A]) = det^2(A)$ is well defined and that ϕ is a fibration. The fiber of ϕ is called the special lagrangian Grassmanian. Compute its dimension. Show, that the special lagrangian Grassmanian is a homogenous space for some connected compact Lie group. Say few words about what you get for $n = 2$.

4 Exercises for the fourth lecture

4.1 Lie algebra is simple if it does not admit any quotient Lie algebra. Show that for a compact simple Lie algebra there exists only one up to a constant invariant scalar product.

4.2 Let $\langle -, - \rangle$ be a G-invariant scalar product (or any symmetric form). Show that $\phi(X, Y, Z) = \langle X, [Y, Z] \rangle$ is an antisymmetric trilinear form.

4.3 Prove that $Der(\mathfrak{sl}_n(\mathbb{R})) \simeq \mathfrak{sl}_n(\mathbb{R})$. (See 2.3 for the definition.)

4.4 Show that $Aut(SL_n(\mathbb{R}))_0 = PSL_n(\mathbb{R})$ and $Out(SL_n(\mathbb{R})) = \pi_0(Aut(SL_n(\mathbb{R}))) = \mathbb{Z}/2$. (At least show that $A \mapsto (A^T)^{-1}$ is not an inner automorphism.)

4.5 ♠ Let $A^* = \overline{A}^T$. Let $G \subset GL_n(\mathbb{C})$ be a complex subgroup which is invariant with respect to the Cartan involution $A \mapsto (A^*)^{-1}$. Define a hermitian product in $\mathfrak{g} \subset M_{n\times n}(\mathbb{C})$ by the formula $\langle\langle X, Y\rangle\rangle = tr(XY^*)$. The hermitian product in $\mathfrak{g}$ allows to define the Cartan involution Θ in $Aut(\mathfrak{g})$. Show that $Ad(G) \subset Aut(\mathfrak{g})$ is Θ-invariant. (Hint: Show that $(ad_X)^* = ad_{X^*}$.)

4.6 ♠ Let G be a connected (compact) semisimple Lie group with an involution $\sigma \in Aut(G)$. Then a symmetric space (of compact type) for G is a homogeneous space G/H where H is an open subgroup of the fixed point set of σ (in other words $G_0^\sigma \subset H \subset G^\sigma$).
– Show, that the following manifolds are symmetric spaces $SU(n)/SO(n)$, $SU(2n)/Sp(n)$, $SU(p+q)/S(U(p)\times U(q))$, $SO(p+q)/(SO(p)\times SO(q))$, $SO(2n)/U(n)$, $Sp(n)/U(n)$, $Sp(p+q)/(Sp(p)\times Sp(q))$.
– Show that σ induces an involution of the symmetric space. For $x = [g] \in G/H$ the composition $L_g \sigma L_g^{-1}$ defines an involution of G/H fixing x; denote it by σ_x. Compute the differential of σ_x at x.
– Show that for any $x, y \in G/H$ there exists z, such that $\sigma_z(x) = y$.

5 For the fifth lecture

5.1 ♠ Compute the Killing form $Tr(ad_X \circ ad_Y)$ for $sl_n(\mathbb{C})$ and $gl_n(\mathbb{C})$. Show, that for $sl_n(C)$ the Killing form is equal up to a constant to $B_0(X, Y) = Tr(XY)$.

5.2 ♠ Let $E \to X$ be a real symplectic vector bundle (ie. the transition functions in some trivialization atlas $\{U_i\}$ consists of $U_i \cap U_j \to Sp(n, \mathbb{R})$ or equivalently there is chosen a symplectic form ω_x in each fiber E_x in a continuous way). Show that E admits a scalar product, such that for $a, b \in E_x$ the endomorphism $J_x : E_x \to E_x$ satisfying $\langle J(a), b\rangle = \omega(a, b)$ is a complex structure (i.e. $J^2 = -1$). Show that any such scalar product can be deformed to a fixed one. (Hint $Sp(n, \mathbb{R})/U(n)$ is contractible.)

5.3 ♠ Show that for any element g of a topological group G $closure\langle g\rangle$ is abelian. For $G = U(n)$ characterize those elements for which $closure\langle g\rangle$ is a n-dimensional torus.

5.4 ♠ Compute what are the maximal tori in $U(n)$, $SU(n)$, $SO(n)$ and $Sp(n)$. What are the normalizers $N(T)$ and the Weyl groups.

5.5 ♠ Show that in $U(n)$ every commutative subgroup is included in a maximal torus. This is not the case eg for $SO(3)$.

5.6 ♠ Let T be a torus (compact connected commutative Lie group). Show that there exists $g \in T$ such that $\langle g\rangle$ is dense in T.

5.7 ♠ Consider the action of $GL_n(\mathbb{K})$ (for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) on $\Lambda^3\mathbb{K}^n$. For which n does there exist an open orbit? What can you say about the stabilizer of an element belonging to an open orbit. At least compute its dimension.

5.8 Check, that Lie algebras $\mathfrak{sl}(n)$, $\mathfrak{so}(n)$ and $\mathfrak{sp}(n)$ are simple.

5.9 Prove that if G is compact, then $(\Lambda^k\mathfrak{g}^*)^G \simeq H^k(G; \mathbb{R})$. (see eg [Wojtyński, Grupy i Algebry Liego]) Is it true for noncompact groups?

6 Exercises for the sixth lecture

6.1 Suppose V, a representation G, can be decomposed as a sum of irreducible representations. Construct a natural isomorphism

$$\bigoplus Hom(V_\alpha, V) \otimes V_\alpha \to V\,,$$

where the summation runs over all isomorphism classes of irreducible representations of G.

6.2 Prove that the category of graded complex vector spaces is equivalent to the category of complex spaces with S^1 action. Both categories admit a monoidal structure ($\otimes$ of graded spaces, and $\otimes$ of representations). Show that these categories are equivalent as monoidal categories.

6.3 Decompose $Hom(V,V)$ into irreducible representations of $G = GL(V)$, where G acts on $Hom(V,V)$ conjugation.

6.4 Show that the natural representation of $SL_2(\mathbb{C})$ is isomorphic to its dual. (This is not true for $GL_2(\mathbb{C})$.)

6.5 Show $\Lambda^{n-1}\mathbb{C}^n = (\mathbb{C}^n)^*$ as representations of $SL_n(\mathbb{C})$.

6.6 Decompose bilinear forms on V into irreducible representations of $GL(V)$.

6.7 Show that irreducible representations of $G \times H$ are of the form $V \otimes W$, where V is a irreducible representation of G and W is a irreducible representation of H.

6.8 For V, a representation of G let us define λV, a representation of $G \times S^1$

$$\lambda V = \bigoplus_{k=0}^{\dim V} \Lambda^k V \otimes \mathbb{C}_1^{\otimes k}\,,$$

where $\mathbb{C}_1$ is the natural representation of $S^1 \subset \mathbb{C}$. Show that $\lambda(V \oplus W) \simeq \lambda V \otimes \lambda W$.

6.9 Let V be irreducible real representation of odd dimension. Show that $V_{\mathbb{C}}$ is irreducible. If the dimension is even it can happen that $V_{\mathbb{C}} \simeq W \oplus \overline{W}$.

6.10 Show that two real representation are isomorphic if and only if their complexification are isomorphic.

7 Seventh lecture

7.1 ♦ Let G be a complex reductive group with Cartan involution Θ. Show that G^Θ is a maximal compact subgroup of G.

7.2 Let $0 \to V_1 \to V_2 \to \cdots \to V_n \to 0$ be an exact sequence of representations. Show that $\sum_{i=1}^n (-1)^i [V_i] = 0 \in R(G)$.

7.3 ♦ Show what are the root spaces in $\mathfrak{sp}(n)$.

7.4 ♦ Show what are the root spaces in $\mathfrak{so}(n)$ for the standard quadratic form $\sum x_i^2$. (Do not change the quadratic form).

7.5 Adams operations [B-tD, p.105]: Assume $k \le n$. Let $Q_k \in \mathbb{Z}[e_1, e_2, \dots, e_n]$ be polynomials satisfying

$$\sum_{i=1}^n x_i^k = Q_k(\sigma_1(x_1,x_2,\dots,x_n), \sigma_2(x_1,x_2,\dots,x_n), \dots, \sigma_n(x_1,x_2,\dots,x_n))$$

where σ_i is the elementary symmetric polynomial. The polynomial Q_k depends only on e_i, $i \le k$ and Q_k does not depend on n, provided, that $n \ge k$. Define a map of the representation ring $\psi^k : R(G) \to R(G)$ by the formula $\psi^k(V) = Q_k(V, \Lambda^2 V, \dots, \Lambda^k(V))$.
– Show that $\chi_{\psi^k(V)}(g) = \chi_V(g^k)$
– Show that $\psi^k \psi^\ell(V) = \psi^{k\ell}(V)$

8 Eighth lecture

Representations of $SL_2(\mathbb{C})$

8.1 Let $v \in V$ satisfy $Hv = nv$ and $Xv = 0$. Compute $X^m Y^m v$.

8.2 Decompose $Sym^3(\mathbb{C}^2) \otimes Sym^2(\mathbb{C}^2)$. Find highest weight vectors for irreducible subrepresentations.

8.3 Find a general formula for multiplicities of irreducible subrepresentations of $Sym^m(\mathbb{C}^2) \otimes Sym^n(\mathbb{C}^2)$.

8.4 Decompose $Sym^2 Sym^3(\mathbb{C}^2)$. Find highest weight vectors for irreducible subrepresentations.

8.5 Show that $Sym^n(Sym^2(\mathbb{C}^2)) \simeq \bigoplus_{s=0}^{[n/2]} Sym^{2n-4s}(\mathbb{C}^2)$.

8.6 What are the irreducible representations of $GL_2(\mathbb{C})$?

8.7 ★ For a Lie algebra $\mathfrak{g}$ denote by $Der(\mathfrak{g})$ the set of linear maps $\phi : \mathfrak{g} \to \mathfrak{g}$ satisfying $\phi([X,Y]) = [\phi(X), Y] + [X, \phi(Y)]$. Show that if $\mathfrak{g}$ is simple, then $Der(\mathfrak{g}) \simeq \mathfrak{g}$.

8.8 Give the precise formula for the action of the Lie algebra $\mathfrak{g}$ on $Hom_G(V,W)$.

8.9 Show that $Sym^k Sym^\ell(\mathbb{C}^2) \simeq Sym^\ell Sym^k(\mathbb{C}^2)$.

9 The 9-th lecture

9.1 Which representations of $SU(2)$ factors through $SO(3)$?

9.2 ♦ Fill the detail of the proof of 9.6 from lecture notes: suppose G is of rank one, then G contains a subgroup isomorphic to $SU(2)$ or $SO(3)$.

9.3 Show that the Heisenberg group N/Z (from short notes from the lectures 1.2) cannot be embedded to a matrix group. (See [Segal, Theorem 6.5])

9.4 ♦ Decompose $Sym^2(\mathbb{C}^3) \otimes (\mathbb{C}^3)^*$ into irreducible representations of $SL_3(\mathbb{C})$.

9.5 Study the examples of $\mathfrak{sl}_3(\mathbb{C})$ representations of Fulton-Harris: exercises §12-§13.

10 Tenth lecture

10.1 ♦ Suppose are given $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_{-\alpha}$. Show that one can rescale X and Y, such that together with $H = [X,Y]$ they satisfy the standard relation of $\mathfrak{sl}(2)$.

10.2 Check that the $e_i - e_j$, $\pm e_i$ in the space $\mathbb{R}^3/lin(e_1 + e_2 + e_3)$ is an abstract system of roots. Compute the group generated by the reflections.

10.3 List all the abstract systems of roots in $\mathbb{R}^2$.

10.4 Let V be an irreducible representation of $\mathfrak{sl}(n)$. Let $v \in V$ be a vector such that $E_{ij}v = 0$ for $i < j$. Show that every wector $w \in V$ can be written as a combination of vectors obtained from v by subsequent aplication of $E_{k+1,k}$ for $k = 1, 2, \ldots n-1$.

10.5 Let G be compact or reductive group. Suppose that the rank of G is equal to 2, $\dim G = 2(n+1)$. Show that there is an exact sequence

$$0 \to \mathbb{Z}_n \to W \to \mathbb{Z}_2 \to 0.$$

10.6 Let G be compact or reductive group. Suppose $rank(G) = k$, $\dim(G) = 3k$, $\dim(Z(G)) = 0$. Then $W = (Z_2)^k$.

10.7 Suppose $R \subset V$ is an abstract root system. Show that $\{\alpha^* = \frac{2}{(\alpha,\alpha)}\alpha \mid \alpha \in R\}$ is an abstract root system.

11 Eleventh

Def: Simple root = undecomposable root = it is not a sum of two positive roots.

11.1 Show that every positive root is a sum of simple roots.

11.2 Simple roots are linearly independent.

11.3 Compute $\beta_0 = \frac{1}{2}\sum_{\alpha\in R_+}\alpha$ for classical Lie algebras and for $\mathfrak{g}_2$.

11.4 Let β_0. Then for a positive root α we have $(\beta_0, \alpha) > 0$.

11.5 The edges of the distinguished Weyl chamber are spanned by vectors from Λ^*_{coroot}. Compute these vectors for classical Lie algebras (find the shortest). For $\mathfrak{sl}(n)$ their sum is equal to β_0. Is it true for other groups?

11.6 Is $\overline{K}_0 \cap \Lambda$ spanned by edge vectors? Check for classical groups.

11.7 Let $\mathfrak{g}$ be a Lie algebra. $U(\mathfrak{g})$ inherits the length filtration from the tensor algebra $T(\mathfrak{g})$. Precisely:

$$F_k U(\mathfrak{g}) = \text{image of} \bigoplus_{0\le i\le k} \mathfrak{g}^{\otimes i}.$$

Show that the graded algebra $Gr^F U(\mathfrak{g})$ is commutative and isomorphic to $\bigoplus_{k=0}^{\infty} Sym^k(\mathfrak{g})$.

11.8 Describe the Verma modules $M(w)$ for $\mathfrak{sl}(2)$. Find a base, the action of X, Y, H. Find the maximal proper submodules $w \in \mathbb{N}$. What breakes down for $w \in \mathbb{R}\setminus\mathbb{N}$.

11.9 Describe the Verma modules for $\mathfrak{sl}(3)$ for the weights L_1 and $3L_1 + L_2$. Find a convenient base. Find the maximal proper submodules. Are they generated by single eigenvectors for $\mathfrak{t}$?

11.10 What are the highest weights of irreducible representations of $\mathfrak{sl}(4)$, $\mathfrak{so}(6)$, $SL(4)$ and $SO(6)$? Find a representation which contains the irreducible representations of a given weight.

11.11 What are the highest weights of irreducible representations of $SO(7)$ and $SO(8)$? Find representations which contain the irreducible representations of a given weight.

12 Twelfth

12.1 Show that for any Lie group $\pi_2(G)$ is trivial.

12.2 Let $G_{\mathbb{C}}$ a connected complex reductive group and G be its maximal compact subgroup. Show that $G/T = G_{\mathbb{C}}/B_+$, where B_+ is the is the connected Lie group with the Lie algebra $\mathfrak{b}_+ = \mathfrak{t}\oplus\bigoplus_{\alpha>0}\mathfrak{g}_\alpha$.

12.3 Compute $\pi_1(SL_n(\mathbb{C}))$ and $Z(SL_n(\mathbb{C}))$ using the lattice in .

12.4 Compute the lattices of roots an coroots for $GL(n)$ (roots will be of smaller rank and Λ_{roots} will not be discrete). Check that the formulas for $Z(G)$ and for $\pi_1(G)$ work.

12.5 Compute everything about the group with the Dynkin diagram G_2: the center, π_1, highest weights irreducible of representations.

12.6 $SL_4(\mathbb{C})$ acts on $\Lambda^2\mathbb{C}^4$ preserving the form $(\xi, \eta) = \xi \wedge \eta$. From this construction find maps $SL_4(\mathbb{R}) \to SO(3,3)$ and $SU(4) \to SO(6)$

12.7 Do the same for $Sp_2(\mathbb{C})$ acting on $ker(\omega : \Lambda^2\mathbb{C}^4 \to \mathbb{C})$.

13 For future

13.1 Show that $\dim V_\lambda = \prod_{1\le i\le j\le n} \frac{\lambda_i-\lambda_j+j-i}{j-i}$

13.2 Check the Weyl character formula for $\lambda = 1^k$ and $\lambda = k$

13.3 Irreducible representations of $GL(n)$ are irreducible as representations of $SL(n)$.

13.4 Show that the Schur function S_λ is a polynomial, and ,,does not depend" on the number of variables, provided that this number is sufficiently large (in the sense, that from the bigger n one passes to smaller by setting $t_i = 0$, and also knowing $S_\lambda(t_1, t_2, \dot{;} t_n)$ for fixed n determines the shape for bigger n if $deg \le n$.

13.5 Check how Weyl character formula works for $sp(n)$.

13.6 Find Kostka numbers of the irreducible representation of $SL_n(\mathbb{C})$ corresponding to the diagram $\lambda = (n-1, n-2, n-3, \dots, 1, 0)$.
,,Bott periodicity" for complex Clifford algebras: Check that $C_{n+2} \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic to the algebra of 2×2 matrices with coefficients in $C_n \otimes_{\mathbb{R}} \mathbb{C}$.

13.7 Compute the group of invertible elements C_2^* of the real Clifford algebra and the Clifford group Γ_2. Which two circles in Γ_2 form $Pin(2)$?

13.8 Find explicit isomorphisms or show that it does not exist between representations of $Spin(n)$:
– spinors S and S^* for n odd
– spinors $S^\pm$ and $(S^\pm)^*$ for n even

13.9 Which spinors are complexifications of real representations of $Spin(n)$? The answer depends on the divisability of n by 8.

13.10 Check the isomorphism of $Spin(2n)$ representations

$$Sym^2(S^+) = (\lambda^n)^+ + \lambda^{n-4} + \lambda^{n-8} + \dots$$

$$\Lambda^2(S^+) = \lambda^{n-2} + \lambda^{n-6} + \lambda^{n-10} + \dots$$

$$Sym^2(S^-) = (\lambda^n)^- + \lambda^{n-4} + \lambda^{n-8} + \dots$$

$$\Lambda^2(S^-) = \lambda^{n-2} + \lambda^{n-6} + \lambda^{n-10} + \dots$$

Here λ^k is the k-th exterior power of the natural representation of the orthogonal group.

13.11 Check the isomorphism of $Spin(2n+1)$ representations

$$Sym^2(S) = \lambda^n + (\lambda^{n-3} + \lambda^{n-4}) + \dots + (\lambda^{n-4i-3} + \lambda^{n-4i-4}) + \dots$$

$$\Lambda^2(S) = (\lambda^(n-1) + \lambda^{n-2}) + (\lambda^{n-5} + \lambda^{n-6}) + \dots + (\lambda^{n-4i-1} + \lambda^{n-4i-2}) + \dots$$

14 Preparatory problems for practical test

14.1 Decompose $\Lambda^4 Sym^4 \mathbb{C}^2$ into irreducible representations of $SL_2(\mathbb{C})$

14.2 Suppose that $H \subset G$ and $rank\, H = rank\, G$. Show that every root of H is a root of G. Give interesting examples ($GL_n(\mathbb{C}) \times GL_m(\mathbb{C}) \subset GL_{m+n}(\mathbb{C})$ is a trivial example). Compute Weyl groups.

14.3 We have an inclusion $GL(2,\mathbb{C}) \hookrightarrow Sp_2(\mathbb{C})$, $A \mapsto \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}$. Which representations of $GL(2,\mathbb{C})$ admit an extended action of $Sp_2(\mathbb{C})$?

14.4 Which representations of $SO(5,\mathbb{C})$ admit an extended action of $SL_5(\mathbb{C})$?

14.5 Which representations of $SL_5(\mathbb{C})$ factor through $PSL_5(\mathbb{C})$. The same question for $Sp_2(\mathbb{C})$ and $Sp_2(\mathbb{C})/center$.

14.6 Let G be a group with the root system of the type G_2. What can be the center and π_1?

14.7 List all the weights of the representation $\Lambda^3 Sym^2\mathbb{C}^3$ of $SL_3(\mathbb{C})$.

14.8 List all the weights of the representation $\Lambda^3\mathbb{C}^6$ of $Sp_3(\mathbb{C})$.

14.9 Decompose into irreducible representations $Sym^2 Sym^2\mathbb{C}^3$ of $SL_3(\mathbb{C})$.

14.10 Show that $\Lambda^k\mathbb{C}^n$ is irreducible as a representation of $SL_n(\mathbb{C})$.

14.11 Decompose into irreducible representation $Sym^4\mathbb{C}^5\otimes\Lambda^4\mathbb{C}^5$ ($SL_5(\mathbb{C})$ is acting). Find the highest weight vectors.

14.12 Decompose into irreducible representation $\Lambda^2\Lambda^3\mathbb{C}^4$ ($SL_4(\mathbb{C})$ is acting).

14.13 List the possible highest weights of the representations of $\mathbb{C}^*\times SL_n(\mathbb{C})$ which come from $GL_n(\mathbb{C})$ via the surgective map $(t,A)\mapsto tA$.

14.14 Let V be the representation of $SL_7(\mathbb{C})$ with highest weight $3L_1+2L_2$. Find the coeficient in the character of the monomial $t_1^2t_2^2t_3$ (the Kostka number $K_{(3,2),(2,2,1)}$).

14.15 Let $V\simeq\mathbb{C}^4$ be the natural representation of $Sp_2(\mathbb{C})$ and $W=\ker(\omega:\Lambda^2V\to\mathbb{C})$. List the weights of the representation $V\otimes Sym^2W$. Is it irreducible?

14.16 Compute $\Lambda_{coroot}/\Lambda_{root}$ for $SO(6)$.

14.17 Let $V\simeq\mathbb{C}^5$ be the natural representation of $SO(5,\mathbb{C})$. Find a nonzero map of representations $Sym^kV\to Sym^{k-2}V$. Is the kernel irreducible?