Positivity

of Schur function expansions

# of Thom polynomials

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joint with

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J^k – space of jets $(C^m, 0) \to (C^n, 0)$

$\Sigma \subset J^k$ – singularity type
= any closed algebraic subset,
invariant w/r to automorphisms

$f : M^m \to N^n$ – holomorphic map
of complex manifolds

Let $\Sigma(f)$ – the set of points in M
of the type Σ

Suppose Σ is closed under suspension.

Thom: There exists a polynomial
$P_\Sigma \in Z[c_1, c_2, c_3, \ldots]$ such that

$$[\Sigma(f)] = P_\Sigma(f^* c_* N / c_* M) \in H^*(M),$$

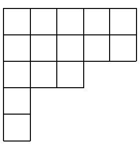
i.e. $[\Sigma(f)] \in H^*(M)$ depends only on Chern classes of M, N and the homotopy class of f.

Schur polynomials

$$Z[c_1, c_2, c_3, \ldots] = \lim_{i,j\to\infty} H^*(\mathrm{Grass}_i(C^j))$$

Schubert cells $\longrightarrow$ basis of $H_*(\mathrm{Grass}_i(C^j))$

Dual basis = Schur polynomials S_I
(for I - Young tableau)

E.g.  $= S_{55311}$

$$= \det \begin{pmatrix} c_5 & c_6 & c_7 & c_8 & c_9 \\ c_4 & c_5 & c_6 & c_7 & c_8 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ 0 & 0 & 1 & c_1 & c_2 \\ 0 & 0 & 0 & 1 & c_1 \end{pmatrix}$$

Theorem: (P. Pragacz, AW) For any singularity type P_Σ is a nonnegative combination of Schur polynomials.

Method of the proof:
– reduction to the universal quotient bundle Q over $\mathrm{Grass}_m(C^j)$
(instead of $T^*M - f^*T^*N$)

$$\widetilde{\Sigma} \subset \tilde{J}^k = \left(\oplus_{\ell=1}^k \mathrm{Sym}^\ell Q\right)^n \quad \downarrow \quad \mathrm{Grass}_m(C^j)$$

$\widetilde{\Sigma}$ – universal singularity locus

Q globally generated

$\Downarrow$

$\tilde{J}^k$ globally generated

– (Fulton-Lazarsfeld)
"positivity of cone classes"

$\Downarrow$

intersection $[\widetilde{\Sigma}] \cdot [\mathrm{Schubert\,cell}] \geq 0$

Fulton–Lazarsfeld

E – ample vector bundle
i.e. $\mathrm{Sym}^i(E)$ for $i >> 0$ has lots of sections

Theorem: If $E \to X$ is an ample bundle $C \subset E$ a cone, $\operatorname{codim} C = \dim X$ then

$$[C] \cdot [X] > 0$$

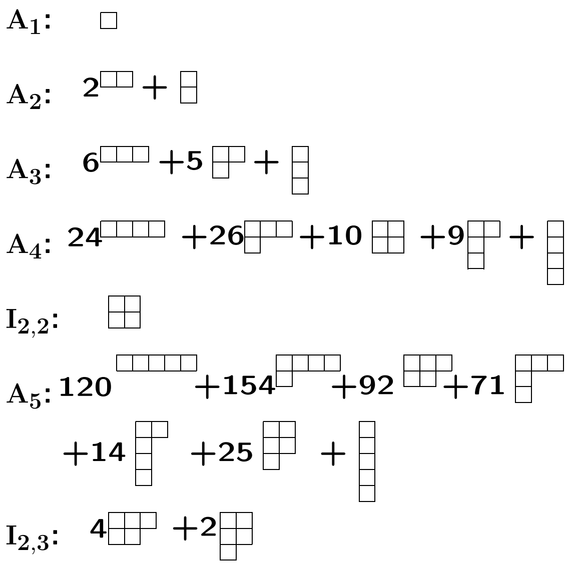
By limit process Theorem applies to globally generated bundles with $\geq$.

Fulton–Lazarsfeld $\Leftarrow$ Bloch–Gieseker

Theorem: If $E \to X$ is an ample bundle $\dim E \geq \dim X = n$ then

$$\int_X c_n(E) > 0$$

Maps $C^n \to C^n$



A_6: $720\,S_6 + 1044\,S_{51} + 770\,S_{42} + 266\,S_{33}$

$+580\,S_{411} + 455\,S_{321} + 70\,S_{222} + 155\,S_{3111}$

$+84\,S_{2211} + 20\,S_{21111} + S_{111111}$

Important in the proof:

Functor

$$Q \mapsto \left(\bigoplus_{\ell=1}^{k} \mathrm{Sym}^\ell Q \right)^n$$

preserves globally generated bundles.

(Any polynomial functor is OK /C)

Another situation: Lagrangian singularities.

Λ_∞ – Lagrangian Grassmannian

$\widetilde{Q}_I$ basis of $H^*(\Lambda_\infty)$ dual to the cell decomposition (Pragacz–Ratajski)

$\widetilde{Q}$-polynomials defined by Pfaffians

$$I = (i_1 > i_2 > \ldots > i_{2k} \geq 0)$$

$$\mathrm{Pf}(\widetilde{Q}_{i_k, j_\ell})$$

where

$$\widetilde{Q}_{ij} = c_i c_j + 2 \sum_{p>0} c_{i+p} c_{j-p} \quad \text{for } i > j$$

Lagrangian singularities - char. classes

Arnold, Fuks, Vassilyev, ... , Kazarian

Theorem:(M. Mikosz, P. Pragacz ,AW) Thom polynomial of any Lagrangian singularity is nonnegative combination of $\widetilde{Q}_I$'s.

Method of the proof:
$\mathcal{L}_n$ – the space parametryzing k-jets of Lagrangian submanifolds

Λ_n – Lagrangian Grassmannian

$\mathcal{L}_n \to \Lambda_n$ – retraction, contractible fibers

Normal bundle of Λ_n in $\mathcal{L}_n$ isomorphic to

$$\bigoplus_{\ell=3}^{k+1} \mathrm{Sym}^\ell(Q)$$

$A_\bullet$-series:

$A_2:\ \widetilde{Q}_1$

$A_3:\ 3\,\widetilde{Q}_2$

$A_4:\ 3\,\widetilde{Q}_{21} + 12\,\widetilde{Q}_3$

$A_5:\ 27\,\widetilde{Q}_{31} + 60\,\widetilde{Q}_4$

$A_6:\ 87\,\widetilde{Q}_{32} + 228\,\widetilde{Q}_{41} + 360\,\widetilde{Q}_5$

$A_7:\ 135\,\widetilde{Q}_{321} + 1275\,\widetilde{Q}_{42} + 2004\,\widetilde{Q}_{51} + 2520\,\widetilde{Q}_6$

$D_\bullet$-series:

$D_4:\ \widetilde{Q}_{21}$

$D_5:\ 6\,\widetilde{Q}_{31}$

$D_6:\ 12(\widetilde{Q}_{32} + 2\,\widetilde{Q}_{41})$

$D_7:\ 24(\widetilde{Q}_{321} + 5\,\widetilde{Q}_{42} + 6\,\widetilde{Q}_{51})$

$E_\bullet$-series:

$E_6:\ 9\,\widetilde{Q}_{32} + 6\,\widetilde{Q}_{41}$

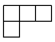
$E_7:\ 9\,\widetilde{Q}_{321} + 60\,\widetilde{Q}_{42} + 24\,\widetilde{Q}_{51}$

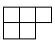
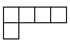
Singularity P_8 (modality 1):

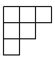
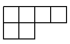
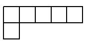
$P_8:\ \widetilde{Q}_{321}$

$D_\bullet$-series:

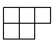
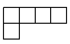
$D_4:$ 

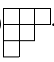
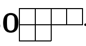
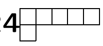
$D_5:$ 6 

$D_6:$ 12( +2 )

$D_7:$ 24( +5  +6 )

$E_\bullet$-series:

$E_6:$ 9  +6 

$E_7:$ 9  +60  +24 

Singularity P_8 (modality 1):

$P_8:$