Lectures on Characteristic Classes of Constructible Functions

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Abstract

The following lectures were delivered at the Mini-School "Characteristic classes of singular varieties" in Banach Center, 23–27 April 2002, by Jörg Schürmann. These lectures discuss the calculus of characteristic classes associated with constructible functions on possibly singular varieties, and focus on the specialization properties. The point of view of characteristic classes of Lagrangian cycles is emphasized. A Verdier-type Riemann-Roch theorem is discussed.

Contents:

1 History

1.1 Characteristic classes of vector bundles

The notion of a characteristic class of a vector bundle has appeared in several contexts.

1.1.1 Chern class of a complex vector bundle

Consider two functors on the category of topological spaces:

[1]The contribution of Piotr Pragacz (partially supported by KBN grant No. 2P03A 024 23) and Andrzej Weber is - apart of some aspects of proofs, mentioned in the text - of purely redactional nature. The text was authorized by J. Schürmann.

- the set of isomorphism classes of complex vector bundles over X,
- the cohomology group of X.

The Chern class is a contravariant functorial class in integral cohomology, i.e. a natural transformation

$$c^* : Vect_{\mathbf{C}}(-) \to H^{2*}(-;\mathbf{Z}),$$

For a complex vector bundle V over X

$$c^*(V) = 1 + c^1(V) + c^2(V) + \ldots + c^{rk(V)}(V), \qquad c^i(V) \in H^{2i}(X;\mathbf{Z}).$$

For a map $f : X' \to X$ one can compute the Chern class of the induced bundle f^*V by the formula:

$$c^*(f^*V) = f^*(c^*(V)) \in H^{2*}(X';\mathbf{Z}).$$

(Cf. [Ch].)

1.1.2 Stiefel-Whitney class of a real vector bundle

It is a contravariant functorial class in mod 2 cohomology, i.e. a natural transformation

$$w^* : Vect_{\mathbf{R}}(-) \to H^*(-;\mathbf{Z}/2).$$

(Cf. [MiSt].)

1.1.3 Euler class of a real oriented vector bundle

It is a contravariant functorial class in integral cohomology, i.e. a natural transformation

$$e : Oriented\, Vect_{\mathbf{R}}(-) \to H^*(-;\mathbf{Z}).$$

For a bundle of rank r over X the Euler class lies in $H^r(X;\mathbf{Z})$.
(Cf. [MiSt].)

1.1.4 Characteristic class of an algebraic vector bundle over an algebraic variety

The Chern classes can be defined as *operators* acting on the Chow groups (Fulton, [Fl, Ch.3]). The action is denoted by $\cap$

$$c^i(V)\cap : A_k(X) \to A_{k-i}(X),$$

$$[\alpha] \mapsto c^i(V) \cap [\alpha].$$

1.2 Properties of characteristic classes of vector bundles

We will concentrate on the case of complex bundles but the formulas hold for real bundles with Chern classes replaced by Stiefel-Whitney classes.

1.2.1 Whitney formula

For a short exact sequence of vector bundles:

$$0 \to V' \to V \to V'' \to 0$$

we have

$$c^*(V) = c^*(V') \cup c^*(V'') \,.$$

Also for a pair of bundles $V \in Vect_{\mathbf{C}}(X)$ and $W \in Vect_{\mathbf{C}}(Y)$

$$c^*(V \times W) = c^*(V) \times c^*(W) \in H^{2*}(X \times Y) \,.$$

1.2.2 Relation with fundamental class

For an algebraic subvariety in a complex variety one can define its fundamental class $Cl(X) = [X]$ in homology with closed supports (so-called Borel-Moore homology). This construction was extended by Borel-Heafliger to a map

$$Cl : A_*(X) \to H^{BM}_{2*}(X; \mathbf{Z}) \,.$$

The Chern classes in $A^*(X)$ and in $H^{2*}(X)$ are related by the formula:

$$Cl(c^i(V) \cap a) = c^i(V) \cap Cl(a) \in H^{BM}_{2(k-i)}(X; \mathbf{Z})$$

for any class of an algebraic cycle $a \in A_k(X)$.

In the real case (by a result of Sullivan, [Su]) one can define a cycle map

$$Cl_{\mathbf{R}} : A_*(X) \to H^{BM}_*(X(\mathbf{R}); \mathbf{Z}/2) \,.$$

Here $X(\mathbf{R})$ denotes the real points of an algebraic variety over $\mathbf{R}$. Then

$$Cl_{\mathbf{R}}(c^i(V) \cap a) = w^i(V(\mathbf{R})) \cap Cl_{\mathbf{R}}(a) \in H^{BM}_{k-i}(X(\mathbf{R}); \mathbf{Z}/2)$$

for any class of a real algebraic cycle $a \in A_k(X)$.

1.2.3 Projective bundle theorem

The following construction shows that the first Chern class is the most important.

Let $\pi : V \to X$ be a vector bundle of rank r. Let $\hat{\pi} : \mathbf{P}(V) \to X$ be its projectivization. As usual $\mathcal{O}(-1) \subset \hat{\pi}^* V$ denotes the universal tautological bundle and $\mathcal{O}(1) = \mathcal{O}(-1)^*$ its dual. Then the cohomology of $\mathbf{P}(V)$ is a free module over $H^*(X)$:

$$H^*(\mathbf{P}(V) = H^*(X)[t]/t^r \,,$$

with $t = c^1(\mathcal{O}(1))$. The algebra structure of $H^*(\mathbf{P}(V))$ determines the Chern classes of V due to the unique relation

$$t^r + c^1(V)t^{r-1} + \ldots + c^r(V) = 0 \,.$$

1.2.4 Grothendieck-Riemann-Roch theorem

The Chern character

$$ch^* : K^0(X) \to H^{2*}(X; \mathbf{Q})$$

is a homomorphism of rings. Here $K^0(X)$ is the Grothendieck group of vector bundles equipped with a ring structure coming from tensor product. We have

$$ch^*([V \otimes W]) = ch^*([V]) \cup ch^*([W]) \,, \qquad ch^*([\mathcal{O}_X]) = 1 \in H^0(X; \mathbf{Q}) \,.$$

The Chern character commutes with pull-backs. Grothendieck-Riemann-Roch theorem describes how the Chern character behaves with respect to a proper push-forward. Let $f : X \to Y$ be a proper map of smooth algebraic varieties. Due to smoothness we can identify K-theory of vector bundles with K-theory of coherent sheaves. Every sheaf has a finite locally free resolution. This is a form of Poincaré duality. We set $K_0(-) = K^0(-)$. We have the induced map

$$f_* : K_0(X) \to K_0(Y)$$

derived from the push-forward of sheaves. On the level of cohomology we have usual Poincaré duality $H^*(-) = H^{BM}_*(-)$ with a switch of degrees. We also have the induced map f_* of homology. Nevertheless the Chern character does not commute with f_* in general. It has to be corrected by the Todd class. The following diagram does commute:

$$\begin{array}{ccccccc} K_0(X) & = & K^0(X) & \xrightarrow{ch^* \cup Td(TX)} & H^*(X; \mathbf{Q}) & = & H^{BM}_*(X; \mathbf{Q}) \\ \downarrow f_* & & & & & & \downarrow f_* \\ K_0(Y) & = & K^0(Y) & \xrightarrow{ch^* \cup Td(TY)} & H^*(Y; \mathbf{Q}) & = & H^{BM}_*(Y; \mathbf{Q}) \end{array} \,.$$

If we take Y to be a point then we obtain the Hirzebruch-Riemann-Roch theorem:

$$\chi(H^*(X, V)) = \int_X ch^*(V) \cup Td(TX) \,.$$

Let us set $Td(X) = Td(TX)$. The Todd class is multiplicative with respect to the Cartesian product:

$$Td(X \times Y) = Td(X) \times Td(Y) \,.$$

This formula can be generalized for singular varieties, see 1.3.5.

1.3 Characteristic classes of singular varieties

The following classes are studied:

Characteristic classes of vector bundles c^*
in cohomology, contravariant functoriality
$\bigcup$
Smooth spaces
$c_*(X) = c^*(TX) \cap [X]$
$\bigcap$
Characteristic classes of singular spaces c_*
in homology, covariant functoriality for proper maps

This picture is contained in a wider context of bivariant theory (Fulton-MacPherson, [FlMa]).

1.3.1

The systematic study of singular spaces started in the sixties. One should mention the work of Thom, Whitney, Hironaka, Łojasiewicz and others on triangulations, stratifications, resolution of singularities (in characteristic zero).

1.3.2

In 1965 M.-H. Schwartz defined certain characteristic classes using obstruction theory of so-called *radial fields*, [Sw]. Her classes were defined for a singular variety X embedded in a smooth manifold M. One obtains Chern-Schwartz classes in $H^{2*}_X(M; \mathbf{Z})$. By Alexander duality this group can be identified with $H^{BM}_{2*}(X; \mathbf{Z})$.

1.3.3

Around 1969 D. Sullivan, [Su], proved that real analytic spaces are mod 2 *Euler spaces* (i.e. the Euler characteristic of the link of any point is even). It follows that the sum of i-simplices in the first barycentric subdivision of any triangulation is mod 2 cycle. This allowed him to define Stiefel-Whitney classes $w_i(X) \in H^{BM}_{\dim X - i}(X; \mathbf{Z}/2)$. For a smooth X these classes are Poincaré dual to the usual ones.

1.3.4

In 1974 R. MacPherson, [Ma], solved the problem of existence of covariantly functorial classes, answering thus a "Deligne-Grothendieck conjecture" (see [Gr, Deuxième Partie, pp.361-368] for an interesting discussion of this conjecture and its general context). MacPherson classes are natural transformations from constructible functions to homology with the normalization condition $c_0(1_X) = [X]$.

There are two ingredients used by MacPherson: *Euler obstruction* and *Chern-Mather classes*. To show that the constructed classes are indeed functorial MacPherson uses his famous *graph construction*. Originally MacPherson classes were constructed for complex algebraic varieties. Later his construction was carried for varieties over arbitrary (algebraically closed) fields of characteristic zero as well as for analytic spaces.

1.3.5

A generalization of Todd classes was studied by Baum-Fulton-MacPherson [BFM]. For a singular variety embedded in a smooth ambient space M one constructs a *localized Chern character* ch_X^M. Then

$$\tau_* = Td(TM) \cap ch_X^M : K_0(X) \to H_{2*}^{BM}(X; \mathbf{Z})$$

is functorial with respect to push-forward and it is independent from the embedding. For a precise construction of ch_X^M as well as for its properties, see [BFM] (compare also with Fulton, [Fl, Ch.18]).

The Todd class of a variety is defined by $Td_*(X) = \tau_*([\mathcal{O}_X])$. It satisfies

$$Td(X \times Y) = Td(X) \times Td(Y)\,.$$

1.3.6

Gonzalez-Sprinberg and Verdier gave an algebraic formula for the Euler obstruction, [G-S].

1.3.7

J.-P. Brasselet and M.-H. Schwartz have shown that MacPherson classes "coincide" with Schwartz classes, [BrSw].

1.3.8

Bivariant theory was developed by Fulton and MacPherson, [FlMa]. Bivariant versions of Stiefel-Whitney classes were given in piecewise linear context as well as a bivariant theory of Todd homology class. Chern classes of *cellular maps* were studied by Brasselet, [Br].

1.3.9

Lagrangian or microlocal approach to Chern and Stiefel-Whitney classes was initiated by M. Kashiwara and his work on the local index theorem for holonomic $\mathcal{D}$-modules, [Ka1] and [Ka2, thm.6.3.1]. A certain topological invariant studied by Kashiwara appeared to be the Euler obstruction (it's a result of Dubson, [Du], compare [BDK, thm.1]). Moreover one should mention the following references and techniques:

- V. Ginzburg: holonomic $\mathcal{D}$-modules for complex algebraic varieties, [Gi];
- C. Sabbah: approach for analytic varieties using geometric methods, [Sa];
- G. Kennedy generalized the construction of Sabbah to algebraic varieties over an arbitrary (algebraically closed) field of characteristic zero, [Ke];
- Kashiwara-Schapira: a microlocal approach to characteristic cycles of constructible sheaves in their book *Sheaves on manifolds*, [KaSp]
- J. Fu: geometric measure theory, [Fu];
- Fu-McCrory: Lagrangian approach to Stiefel-Whitney classes for real algebraic varieties, [FuMc].

1.4

In the remaining lectures we will use the language of the book of Goresky and MacPherson "Stratified Morse theory", [GoMa], to develop a Morse theory for constructible functions (following [GrMa]). For a more sophisticated Morse theory of constructible sheaves one should compare with [Sch1] and [KaSp].

2 Calculus of constructible functions

2.1 Motivation

The motivation for this lecture is the singular Grothendieck-Riemann-Roch theorem [BFM], which states that for a singular subvariety X embedded in a smooth ambient space the map

$$\tau_* = Td(TM) \cap ch_X^M : K_0(X) \to H_*^{BM}(X; \mathbf{Q})$$

is well defined and it is functorial with respect to push-forward. Moreover it is independent from the embedding. The topological counterpart would be a localized class $?_X^M$ making the following diagram commutative

$$\begin{array}{ccc} K_0(X) & \xrightarrow{c^*(TM)\cap ?_X^M} & H_*^{BM}(X;\mathbf{Z}) \\ {\scriptstyle \chi_X} \searrow & & \nearrow {\scriptstyle c_*} \\ & CF(X). & \end{array}$$

Here $CF(X)$ is the group of constructible functions on X. The map χ_X from the Grothendieck group of constructible sheaves is taking the Euler characteristic stalkwise (it is surjective) and c_* is Chern-MacPherson transformation.

2.2 Category

We will work in the following categories

$$\begin{array}{c} \text{algebraic varieties over a field of characteristic zero} \\ \bigcup \\ \text{algebraic complex varieties} \\ \bigcap \\ \text{complex analytic varieties} \\ \bigcup \\ \text{complex analytic varieties with a fixed stratification} \end{array}$$

We will use techniques developed for real sub- and semi-analytic varieties.

2.3 Definitions of constructible functions

2.3.1

Let X be an algebraic variety defined over an algebraically closed field k with $char(k) = 0$. A *constructible set* is a subset defined by finitely many unions, intersections, and complements of (closed) subvarieties. A *constructible function* $\alpha : X \to \mathbf{Z}$ is a function such that $\alpha^{-1}(m)$ is constructible for any $m \in \mathbf{Z}$ and non-empty only for finitely many $m \in \mathbf{Z}$. The set of constructible functions is a group denoted by $CF(X)$. The assumption $char(k) = 0$ will only be used later on (in the definition of or) for the properties of some transformations, e.g. for Theorem 2.7 and Proposition 2.8 in the algebraic context (where one uses l-adique cohomology for the definition of the Euler characteristic).

Each function can be written as a finite sum $\alpha = \sum 1_{Z_i}$, where Z_i are (closed) subvarieties of X.

2.3.2

For a field k of characteristic zero which is not algebraically closed we define

$$CF(X) = CF(X(\bar{k}))^{Gal(\bar{k}/k)}\,.$$

2.3.3

For complex analytic varieties we consider functions which are locally finite sum of the form $\alpha = \sum 1_{Z_i}$ with Z_i being closed analytic subvarieties. Similarly constructible functions are defined for real analytic, subanalytic etc. varieties.

2.4 Transformation of constructible functions

2.4.1 Contravariant functoriality

Let $f : X \to Y$ be a map of varieties. Contravariant transformation

$$f^* : CF(Y) \to CF(X)$$

is just the composition

$$f^*(\alpha) = \alpha \circ f\,.$$

2.4.2 Multiplication

We can multiply functions pointwise

$$(\alpha \cdot \beta)(x) = \alpha(x)\beta(x)\,.$$

2.4.3 Exterior product

We have an operation

$$\times : CF(X) \times CF(Y) \to CF(X \times Y)\,,$$

$$(\alpha \times \beta)(x, y) = \alpha(x)\beta(y)\,.$$

2.4.4 Euler characteristic

We work here in the real world. Denote by $CF_c(X)$ the group of constructible functions with compact support. (By the definition the support is always closed.) Euler characteristic is the unique linear map

$$\chi : CF_c(X) \to \mathbf{Z}\,,$$

$$\chi(1_Z) = \chi(H^*(Z))$$

for Z compact subanalytic. We introduce the following notation for $\alpha \in CF(X)$ and subsets $A \subset B \subset X$, with B compact

$$\chi(B, A; \alpha) := \chi(\alpha \cdot 1_{B\setminus A})\,.$$

We can think about $\chi(B, A; \alpha)$ as a relative homology of the pair (B, A) with coefficients in α. We set $\chi(B; \alpha) = \chi(B, \emptyset; \alpha)$.

Example 2.1 Let $\alpha = 1_Z$. Then

$$\begin{aligned} \chi(B, A; 1_Z) &= \chi(1_Z(1_B - 1_A)) \\ &= \chi(1_{Z\cap B}) - \chi(1_{Z\cap A}) \\ &= \chi(H^*(Z \cap B)) - \chi(H^*(Z \cap A)) \\ &= \chi(H^*_c(Z \cap (B \setminus A)) \\ &=: \chi_c(1_{Z\cap(B\setminus A}) \end{aligned}$$

The last term is called the *Euler characteristic with compact support.*

Example 2.2 Let Z be an open ball in $\mathbf{R}^n$. Then $\chi_c(1_Z) = (-1)^n$.

Euler characteristic with compact support is additive. Suppose a compact set X is a finite disjoint union of locally closed subsets X_i. Then

$$\chi(X; \alpha) = \sum \chi_c(X_i; \alpha)\,.$$

In the complex case we have [2]:

PROPOSITION 2.3 *Let X be a compact (or compactifiable) complex variety decomposed into a disjoint union of locally closed subvarieties X_i. Then for every constructible function α,*

$$\chi(X; \alpha) = \sum \chi(X_i; \alpha)\,.$$

[2] This paragraph is added by PP-AW.

One can propose the following reasoning to prove the above assertion. It is enough to take $\alpha = 1_X$ and $\{X_i\}$ to be the strata of a Whitney stratification of X with $\chi(Z) := \chi(H^*(Z))$ for $Z \subset X$ locally closed.

We record the following result of Sullivan: if L is a compact real Whitney stratified space with strata of even codimension, and dimension of L is odd, then $\chi(L) = 0$. (Cf. [Su].)

We will now show that $\chi(X) = \sum \chi(X_i)$ by induction on the depth $d(X)$. We recall that *depth* is the maximal (real) codimension of strata.

If $d(X) = 0$, then X has a unique stratum, and the assertion is obvious.

Suppose that $d(X) = 2d$. Let X_g be the deepest stratum. This stratum has a neighbourhood U, which is a fibered over X_g with the fiber being the cone over the link L_g (cf. [Mat]). Since L_g is compact of dimension $2d-1$, and the strata are of even codimension (as in X), we have $\chi(L_g) = 0$. Finally,

$$\chi(X) = \chi(X \setminus X_g) + \chi(U) - \chi(U \setminus X_g)$$

$$= \sum_{i \neq g} \chi(X_i) + \chi(X_g) - \chi(L_g)\chi(X_g) = \sum_i \chi(X_i)\,.$$

This proves Proposition 2.3. Of course, if the deepest stratum is not connected, the argument should be slightly modified.

Corollary 2.4 *If X is a complex algebraic variety (or just admitting a compactification stratified by even codimension strata), then $\chi_c(X) = \chi(X)$.*

Corollary 2.4 for a complex algebraic variety is a special case of [Sch1, formula (6.42), p.413] or [La], and compare with 2.4.6.

The following example will be invoked several times.

Example 2.5 Let $a < b < c \in \mathbf{R}$ and α_-, α_0 and α_+ be integers. Define a constructible function on $\mathbf{R}$

$$\alpha = \alpha_- 1_{[a,b[} + \alpha_0 1_{\{b\}} + \alpha_+ 1_{]b,c]}\,.$$

Then

$$\chi([a,c];\alpha) = (\alpha_- - \alpha_-) + \alpha_0 + (\alpha_+ - \alpha_+) = \alpha_0\,.$$

We call this property the *conic structure* (of the interval).

2.4.5 Proper push-forward

Let $f : X \to Y$ be a proper map. We define a transformation of constructible functions

$$f_* : CF(X) \to CF(Y)\,,$$

$$f_*(\alpha)(y) = \chi(f^{-1}(y);\alpha)\,.$$

(It is enough to assume that f is proper on the support of α.) In particular

$$f_*(1_Z)(y) = \chi(f^{-1}(y) \cap Z)\,.$$

Example 2.6 If Y is a point, then $f_*(\alpha) = \chi(\alpha) 1_{pt}$.

THEOREM 2.7 *The proper push-forward of a constructible function is constructible.*

PROPOSITION 2.8 *For a sequence of proper maps* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *we have*

$$(g \circ f)_* = g_* \circ f_* \,.$$

The proofs of these results rely on the stratification properties of maps.

Consider a real-valued function $f : X \to \mathbf{R}^{\geq 0}$ which is proper and a constructible function $\alpha \in CF(X)$. Let $X_0 = f^{-1}(0)$. By 2.7 the function $f_*(\alpha)$ is constructible. It follows that there exists $\epsilon > 0$, such that $f_*(\alpha)_{|]0,\epsilon]}$ is constant. By the *conic structure* of the interval (see Example 2.5)

$$\chi([0,\epsilon]; f_*\alpha) = f_*(\alpha)(0) = \chi(f^{-1}(0); \alpha) = \chi(X_0; \alpha) \,.$$

2.4.6 Verdier duality for constructible functions

We work in the real world. The duality map

$$D : CF(X) \to CF(X)$$

is defined by

$$(D\alpha)(x) = \alpha(x) - \chi(S_\epsilon(x); \alpha) \,,$$

where $S_\epsilon(x)$ is a small sphere centered at x. One can show that $D \circ D = Id$. If we deal with complex varieties then due to Sullivan, [Su], we know that $\chi(S_\epsilon(x); \alpha) = 0$. Thus $D = Id$. (For real varieties we only have $\chi(S_\epsilon(x); \alpha) = 0$ mod 2.) In particular, in the complex algebraic context one gets $\chi = \chi_c$, since in general $\chi \circ D = \chi_c$ in the semi-algebraic context.

2.4.7 Nearby cycles

Let $k = \mathbf{R}$ or $\mathbf{C}$. Let $f : X \to k$ be a function. Set $X_0 = f^{-1}(0)$. In the complex case $k = \mathbf{C}$ we have a Milnor fibration for appropriate choice of $0 < \epsilon \ll \delta \ll 1$ with the fiber

$$M_{f,x} = B_\delta(x) \cap \{f = \epsilon\} \cap X \,.$$

The nearby cycle operation

$$\psi_f : CF(X) \to CF(X_0)$$

is defined by

$$\psi_f(\alpha)(x) = \chi(M_{f,x}; \alpha) \,.$$

The value of $\psi_f(\alpha)$ depends only on the values of α outside of X_0.

In the real world we have two nearby cycle operations (studied by McCrory and Parusiński, [McPa1]): the lower ψ_f^- and the upper ψ_f^+, due to the possibility of taking negative values of ϵ. Negative and positive Milnor fibers might differ.

2.4.8 Vanishing cycles

Let us concentrate on the complex case. The vanishing cycle operation

$$\phi_f : CF(X) \to CF(X_0)$$

is defined by

$$\phi_f(\alpha) = \psi_f(\alpha) - \alpha_{|X_0} \,.$$

The motivation comes from the world of derived category of sheaves. The analogous operations form a distinguished triangle

$$\begin{array}{ccc} i^*F & \longrightarrow & \psi_f F\,. \\ {\scriptstyle +1}\nwarrow & & \swarrow \\ & \phi_f F & \end{array}$$

where i denotes the inclusion of X_0 into X.

Remark 2.9 On the $\mathcal{D}$-module level $\phi_f[-1]$ corresponds to vanishing cycles.

2.4.9 Specialization

The specialization of Verdier [Ve2], has also its counterpart for constructible functions. Let $X \subset Y$ be a closed subset. The specialization is an operation

$$sp : CF(Y) \to CF_{mon}(C_X Y)\,.$$

Here $C_X Y$ denotes the normal cone of X in Y. The subscript *mon* stands for *monodromic* i.e. conic function. Let A^1 be the affine line and $A^* = A^1 \setminus \{0\}$. The normal cone is contained in the deformation space M (cf. [Fl, Ch.5]).

$$M = Bl_{X\times\{0\}}(Y \times A^1) \setminus Bl_{X\times\{0\}}(Y \times \{0\})\,,$$

$$\begin{array}{ccccc} C_X Y & \subset & M & \supset & Y \times A^* \\ \downarrow & & \downarrow g & & \downarrow \\ \{0\} & \subset & A^1 & \supset & A^* \end{array}$$

Specialization is defined by the formula

$$sp(\alpha) = \psi_g(\tilde{\pi}^*(\alpha))\,,$$

where $\tilde{\pi}$ is the composition of the blow up $M \to Y \times A^1$ and the projection to Y.

The value of $sp(\alpha)$ depends only on the values of $\tilde{\pi}^*(\alpha)$ outside of $C_X Y$ and $sp(\alpha)_{|X} = \alpha_{|X}$. Here we consider X as embedded in $C_X Y$ as the vertex-section.

For some aspects of operations on constructible functions, cf. [Sp].

3 Stratified Morse theory for constructible functions and Lagrangian cycles

In this lecture, we work in the real world (following [GrMa]). By a dimension we mean the real dimension.

3.1 Stratification

Let X be a closed subset of a smooth manifold M. We can work over $\mathbf{C}$ as well as over $\mathbf{R}$. A stratification of X is a filtration

$$X_\bullet : \emptyset = X_{-1} \subset X_0 \subset \cdots \subset X_n = X \,,$$

where each inclusion is assumed to be a closed embedding in the appropriate category. Moreover each difference $X^i := X_i \setminus X_{i-1}$ should be a smooth manifold of dimension i or empty. A stratum of $X_\bullet$ is a connected component of some X^i.

The group of constructible functions with respect to a stratification $CF(X_\bullet)$ consists of functions constant on strata. The group of all constructible functions is the direct limit of the groups $CF(X_\bullet)$.

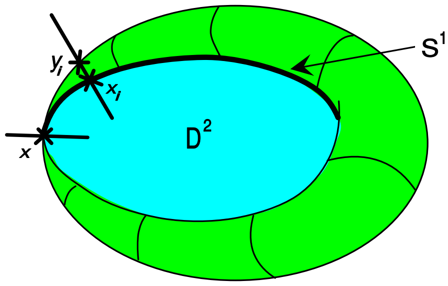
3.2 Whitney conditions

Whitney conditions are traditionally called 'a' and 'b' (although 'b' is stronger then 'a'). Consider the following situation for two strata S_α and S_β. Suppose that there are given sequences of points $x_i \in S_\alpha$ and $y_i \in S_\beta$, both converging to $x \in S_\alpha \cap \bar{S}_\beta$. Assume that the secant line $\overline{x_i y_i}$ converges in the projective space to ℓ and that the tangent space $T_{y_i} S_\beta$ converges to a space V in an appropriate Grassmannian. Then

a) $T_x S_\alpha \subset V$,

b) $\ell \subset V$.

Example 3.1 There are three strata: circle S^1, open disk D^2 and $X \setminus \overline{D}^2$. The Whitney condition b) is not satisfied.



Perhaps, the simplest counterexample for the Whitney condition b) is the "Whitney cusp" $X := \{x^2 + z^2 \cdot (z - y^2)\} \subset \mathbf{R}^3$, with $X_1 := \{x = 0 = z\}$ and $X_2 := X$. It is a-regular, but not b-regular in the point 0. Consider a sequence $(c_n \neq 0)$ coverging to 0, and look at $x_n := (0, c_n, 0) \in X^1 = X_1$ and $y_n := (0, c_n, c_n^2) \in X^2 = X \backslash X_1$. Then $T_{y_n} X^2 = (0, -2c_n, 1)^\perp \to (0, 0, 1)^\perp =: V$, but the corresponding secant line converges to $l = \{x = y = 0\}$, which is orthogonal to $V = \{z = 0\}$.

Let

$$\Lambda := \bigsqcup T_S^* M \subset T^* M$$

be the disjoint union of the conormal spaces to the strata. The Whitney condition a) is equivalent to the statement that Λ is closed in T^*M.

The following is an exercise for using the condition b). Let $S_\alpha = \{x_0\}$. In some local coordinates $x_0 = 0$. Let $\nu = \sum x_i^2$ be the distance function.

PROPOSITION 3.2 *If the Whitney condition b) is satisfied at x_0, then the restriction of ν to each stratum has no critical points near x_0.*

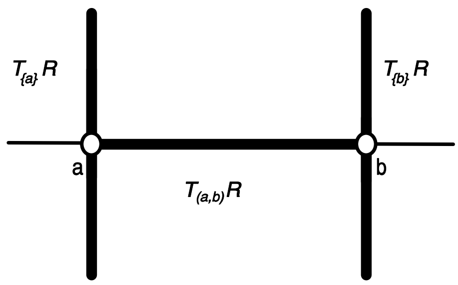
Proof. Assume the converse. Then there exists a sequence $y_i \in S_\beta$, converging to x_0, such that $T_{y_i} S_\beta \subset ker\, d\nu$. The secant $\overline{x_0 y_i} \perp ker\, d\nu$. We can assume that $\overline{x_0 y_i}$ converges to a line ℓ and $T_{y_i} S_\beta$ converges to a space V. Then $\ell \perp V$. This contradicts b).

DEFINITION 3.3 In the conormal space Λ we distinguish the subspace of *good covectors*

$$\Lambda_S^o := T_S^* M \setminus \bigcup_{S' \neq S} \overline{T_{S'}^* M}$$

$$\Lambda^o := \bigsqcup_S \Lambda_S^o .$$

Example 3.4 Let $X = [a, b] \subset \mathbf{R}$ with the stratification $X_0 = \{a, b\} \subset X_1 = X$.



3.3 Morse theory

We will formulate results coming form Morse theory which we will treat as a "black box" here.

By the *normal slice* of a stratum S we mean a germ of a submanifold $N \subset M$ which is transverse to S and $\dim N = \operatorname{codim} S$. Suppose $N \cap S = \{x\}$. Fix a real function with a good differential df_x at x. Define the upper/lower half-link of f by

$$L_f^{\pm} = N \cap X \cap B_\delta(x) \cap \{f = f(x) \pm \epsilon\}\,,$$

for $0 < \epsilon \ll \delta \ll 1$. Fix a constructible function $\alpha \in CF(X_\bullet)$. For each $x \in S$ we will define a *normal index*

$$i(df_x; \alpha) = \alpha(x) - \chi(L_f^-; \alpha)\,.$$

In the complex case we can rewrite this quantity as

$$\alpha(x) - \psi_{g_{|N}}(x) = -\phi_{g_{|N}}(x)\,.$$

Here g is a holomorphic function with $g(x) = 0$ and the real part $f := re(g)$ as before. Moreover it is related to the Euler characteristic of the complex link, [GoMa].

The following facts follow from the Morse theory:

- $i(df_x; \alpha)$ depends only on $df_x \in \Lambda^o$ and it is locally constant on Λ^o,
- Duality: $i(df_x; D\alpha) = (-1)^{\dim S} i(-df_x; \alpha)$ where D is Verdier duality 2.4.6. In particular for complex varieties $i(df_x; D\alpha) = i(df_x; \alpha) = i(-df_x, \alpha)$,
- $i(df_x \times dg_y; \alpha \times \beta) = i(df_x; \alpha) \cdot i(dg_y; \beta)$,
- Change of Euler characteristic is equal to the index. Suppose $f : M \to \mathbf{R}$ is a function such that f has at most one critical point in $f^{-1}[a, b]$ at x. Assume $a < f(x) < b$. Then for any constructible function $\alpha \in CF(X_\bullet)$

  $$\chi(\{a \le f \le b\}, \{f = a\}; \alpha) = \\ = \begin{cases} 0 & \text{if } x \text{ is not a critical point,} \\ (-1)^\lambda i(df_x; \alpha) & \text{if } x \text{ is a stratified Morse critical point.} \end{cases}$$

  Here λ is the Morse index of $f_{|S}$.

Remark 3.5 We say that x is a stratified Morse critical point if df_x is good and $f_{|S}$ is a classical Morse critical point. Then $df(M) \cap \Lambda \subset \Lambda^o$ near df_x and $df(M)$ intersects transversaly $T^*_S M$ at df_x. The right choice of orientation for $T^*_S M$ allows one to express $(-1)^\lambda$ as the intersection number $\sharp[df(M)] \cap [T^*_S M]$.

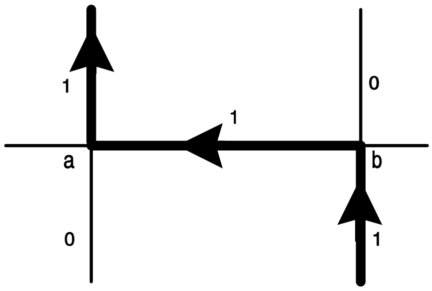
3.4 Lagrangian cycles

The references for this subsection are [GrMa], [ScVi], and [Sa]. For simplicity let's assume that the ambient space M is oriented. The space of good covectors decomposes into the connected components

$$\Lambda^o = \bigsqcup \Lambda^o_j \,.$$

Each component has its index $i(\Lambda^o_j;\alpha)$.

Example 3.6 The indices in Example 3.4 are



We claim:

PROPOSITION 3.7 *The sum* $\sum i(\Lambda^o_j;\alpha)\Lambda^o_j$ *is a homology cycle.*

Remark 3.8 The conormal space $T^*_S M$ can always be oriented. Unfortunately there are several conventions to do this. In the complex case our orientation differs from the complex one by $(-1)^{\dim_{\mathbf{C}} S}$. Moreover, in the complex case, the fundamental class $[\overline{T^*_S M}]$ is already a homology cycle. So Proposition 3.7 is non-trivial only in the real context. Here it goes back originally to Kashiwara (and also to Fu).

Our construction gives us an assignment called *the characteristic cycle* or *microlocal Euler class*

$$CC = \mu eu : CF(X_\bullet) \longrightarrow H^{BM}_{top}(\Lambda) = L(X_\bullet, M)\,.$$

In our notation $L(X_\bullet, M)$ denotes the group of Lagrangian cycles supported on the covectors of the conormal spaces to the strata of $X_\bullet$. One can directly show by induction on $n = dim(X)$, that the map CC is an isomorphism. Here one only uses $i(df_x;\alpha) = \alpha(x)$ for x in a top-dimensional stratum (so that $L^-_f = \emptyset$).

THEOREM 3.9 *There exists an inverse transformation to CC*

$$Eu^\vee : L(X_\bullet, M) \to CF(X_\bullet)\,.$$

It is given by

$$L \mapsto \alpha\,,$$

where the value of α *at* $x_0 \in X$ *is given by the intersection number*

$$\alpha(x_0) = \sharp_{d\nu_0}[d\nu(M)] \cap [L]\,.$$

Here, as before, $\nu(x) = \sum x_i^2$ *in some local coordinates in which* $x_0 = 0$*.*

The map $Eu^\vee$ is in the complex context induced by the local Euler obstruction - cf. Remark 4.5; this motivates the notation.

The theorem follows from the formula (compare example 4.4):

$$\sharp_{d\nu_0}[d\nu(M)] \cap [CC(\alpha)] = \alpha(x_0)\,.$$

Consider now the complex context. If $CC(\alpha) = \overline{T^*_S M}$, that is $\alpha = Eu^\vee(\overline{T^*_S M})$, then one can recover the values of the Euler obstruction of $\bar{S}$:

$$Eu_{\bar{S}}(x_0) = (-1)^{\dim_{\mathbf{C}} S} Eu^\vee_{\bar{S}}(x_0) = (-1)^{\dim_{\mathbf{C}} S} \alpha(x_0)\,.$$

The sign comes from a different choice of orientation when comparing with [Ma].

4 Characteristic classes of Lagrangian cycles

4.1 Recollection of the previous lecture

Let $X_\bullet$ be a Whitney stratified space contained in an oriented smooth manifold M. We use stratified Morse theory for constructible functions as a "black box". Let

$$CC : CF(X_\bullet) \to H^{BM}_{\dim M}(\Lambda) = L(X_\bullet, M)$$

be the characteristic cycle map. (As before $\Lambda = \bigcup_S T^*_S M$ is a closed subspace of T^*M.) Recall that df_x has a stratified Morse critical point on S if and only if in the neighbourhood of df_x

$$df(M) \pitchfork (T^*_S M)^o = \{df_x\}$$

and df_x is good, see Def. 3.3. Then

$$\sharp_{df_x}[df(M)] \cap [T^*_S M] = (-1)^\lambda \,,$$

where λ is the Morse index of $f|S$ in x. If M and S are complex manifolds, then the canonical orientation of $T^*_S M$ differs from the complex orientation and

$$[T^*_S M] = (-1)^{\dim S}[(T^*_S M)_{\mathbf{C}-\text{orientation}}] \,.$$

Remark 4.1 The Lagrangian cycle $CC(\alpha)$ does not depend on the choice of a stratification in which α is constructible. If one subdivides a stratification, then the indices of the additional strata vanish.

Now consider the group $L(T^*M_{|X})$ of all Lagrangian cycles in $T^*M_{|X}$, which by definition is the direct limit of the groups $L(X_\bullet, M)$. In the complex case it is generated by $[\overline{T^*_{S_{\text{reg}}} M}]$, where S is a subvariety of X. These cycles are conic.

Warning. If we work in the real subanalytic context, then *conic* means invariant with respect to the multiplicative action of $\mathbf{R}^+$, not $\mathbf{R}^*$.

Exercise: Show that $CC(\alpha) \subset T^*_M M$ if and only if α is locally constant. Recall that $T^*_M M$ is the zero section of T^*M.

4.2 Intersection formula

THEOREM 4.2 *Suppose $f : M \to [a, \infty[$ is a real analytic function, and $\alpha \in CF(M)$ is a constructible function, such that $f_{|supp(\alpha)}$ is proper and*

$$\pi(supp\, CC(\alpha) \cap df(M)) \subset \{a \leq f \leq b\} \,.$$

Here $\pi : T^*M \to M$ is the projection. Then

$$\sharp[df(M)] \cap [CC(\alpha)] = \chi(\{f \leq c\}; \alpha)$$

for any $c > b$.

For the proof of the theorem one deforms f to a Morse function. By additivity of the Euler characteristic with compact support, one reduces to the case of exactly one Morse critical point $x \in supp(\alpha)$. Then the claim follows from the last result of our "black box" about Morse theory.

Example 4.3 If f is a constant function and $\pi(supp(CC(\alpha)) \cap T^*_M M)$ is compact (this is so when *supp* is compact), then

$$\chi(M;\alpha) = \sharp[T^*_M M] \cap [CC(\alpha)] \,.$$

If $\alpha = 1_M$ for a compact oriented manifold M, the Theorem 4.2 reduces to the well known Poincaré-Hopf theorem. We can also write

$$\chi(X) = \sharp(s^! CC(1_X)) \,,$$

where $s : M \to T^*M$ is the zero section. The last formula can be decomposed into steps:

- $CC(1_X)$ is the microlocal Euler class,
- $s^! CC(1_X)$ is the Euler class,
- $\sharp s^! CC(1_X)$ is the degree of the Euler class i.e. the Euler characteristic.

Example 4.4 Let ν be the function as in Proposition 3.2

$$\nu = \sum x_i^2 : \{\nu < \epsilon\} \to [0, \epsilon[\,.$$

Then

$$\sharp_{d\nu_0}[d\nu(M)] \cap [CC(\alpha)] = \chi(\{\nu \leq \epsilon\}; \alpha) \,.$$

Due to conic structure the latter is equal to

$$\chi(\{\nu = 0\}; \alpha) = \alpha(0)$$

for $0 < \epsilon \ll 1$. This formula gives us an inverse Eu^v to the characteristic cycle map $CC : CF(X) \to L(T^*M_{|X})$.

Remark 4.5 In the complex category the value of $Eu^\vee[\overline{T^*_{S_{reg}} M}] \in CF(X)$ at a point x is just the Euler obstruction $Eu_S(x)$ corrected by the sign $(-1)^{\dim S}$, see 3.9.

4.3 Variety of categories

In other categories our approach works as well. We obtain a map

$$Eu^\vee : L(T^*M_{|X}) \to \mathcal{H}_*(X) \,,$$

where $\mathcal{H}_*(-)$ may denote:

- $A_*(-/k)$ – group of algebraic cycles if we work in the algebraic category of varieties over a field k with $char(k) = 0$. Here one associates to $[\overline{T^*_{S_{reg}}M}]$ (for a subvariety S) the dual Euler obstruction of S using the algebraic definition of [G-S] as in 1.3.6.
- $H^{BM}_{2*}(-)$ – in the analytic or algebraic category over $\mathbf{C}$,
- $H^{BM}_*(X(\mathbf{R}); \mathbf{Z}/2)$ – in the algebraic category over $\mathbf{R}$.

In the last case we obtain Stiefel-Whitney classes. With a suitable modification an Euler class can be obtained in the category of real oriented varieties.

4.4 Segre classes

In the complex category we work with conic cycles which are invariant with respect to $\mathbf{C}^*$. It is convenient to pass to the projectivization of T^*M. With a conic cycle ξ one can associate its Segre class

$$s_*(\xi) = \hat{\pi}_*(c(\mathcal{O}(-1))^{-1} \cap [\hat{\xi}]) \in \mathcal{H}_*(X)\,.$$

Here $\hat{\pi} : \mathbf{P}(T^*M) \to X$ is the projection and $\hat{\xi}$ denotes the projectivization of ξ. If ξ has components in the zero section, then one should use this definition for the projective completion of ξ in $P(T^*M \oplus 1)$.

Remark 4.6 For real algebraic varieties the above construction works since the conic cycles are invariant with respect to $\mathbf{R}^*$. On the level of constructible functions it means that $D(\alpha) = \alpha \bmod 2$. This is just the condition for being an Euler space. See [FuMc].

Remark 4.7 In [Fl], the Segre class of a subscheme of M is defined by a formula similar to the one given above, involving the normal cone of the subscheme in M.

If X is a hypersurface in M, then $s_*CC(1_X)$ can be expressed in terms of the Segre class of X in M and of a correction term, which amounts essentially to the Segre class of the Jacobian subscheme of X in M, cf. e.g. [Al1] or Aluffi's lectures in the present volume.

4.5 Definition of Chern-MacPherson classes

We will introduce dual Chern classes of a constructible function. They are related to the usual Chern-MacPherson classes via the formula

$$c_i^\vee = (-1)^i c_i\,.$$

We remark that the involution

$$\vee : A_*(-) \to A_*(-)\,,$$

$$\vee : H_{2*}(-) \to H_{2*}(-)$$

is useful in various contexts. The classes $c_*^\vee$ are defined by the commuting diagram

$$\begin{array}{ccc} CF(X) & \begin{array}{c} \xrightarrow{CC} \\ \xleftarrow[Eu^\vee]{} \end{array} & L(T^*M_{|X})\,. \\ {\scriptstyle c_*^\vee} \searrow & & \swarrow {\scriptstyle c^*(T^*M)\cap s_*} \\ & \mathcal{H}_*(X) & \end{array}$$

The class $?_X^M := s_*(CC(\cdot))$ (cf. 2.1) resembles the localized Chern character in singular Riemann-Roch theorem of Baum-Fulton-MacPherson, cf. 1.3.5.

4.6 Proper push-forward

In order to prove functoriality of c_* we will define *the proper push-forward* of Lagrangian cycles. Let $f : X \to Y$ be a proper map which extends to a map of ambient spaces, also denoted by $f : M \to N$. (We can even assume that f is submersion. Just replace M by $M \times N$ which contains M as the graph of f.) We have a diagram of bundles

$$\begin{array}{ccccc} T^*M & \xleftarrow{df} & f^*T^*N & \xrightarrow{\tau} & T^*N \\ & \searrow \swarrow & & & \downarrow \\ & M & \xrightarrow{\qquad f \qquad} & & N\,. \end{array}$$

Definition 4.8 A Lagrangian cycle ξ supported over X is transformed by f to

$$f_*([\xi]) := \tau_*(df^*[\xi])\,.$$

Here one uses Poincaré duality for the definition of df^*. That $f_*([\xi])$ really defines a Lagrangian cycle follows from a suitable Whitney stratification of the map $f : X \to Y$ (or from generic smoothness in the algebraic context over a field over characteristic zero).

We will show that the proper push-forward of Lagrangian cycles agrees with the proper push-forward of constructible functions. It is enough to check that

$$Eu^\vee f_* CC(\alpha) = f_*(\alpha)\,.$$

We compute

$$Eu^\vee f_* CC(\alpha)(y) = Eu^\vee(\tau_* df^*[CC(\alpha)])(y)$$

for each $y \in Y \subset N$. By the definition (in Theorem 3.9) of $Eu^\vee$ it is equal to

$$\sharp_{d\nu_y}([d\nu(N)] \cap \tau_* df^*[CC(\alpha)])\,.$$

Let $a_\nu : M \to f^*T^*N$ be the section induced by $d\nu$. We rewrite the last expression as:

$$\sharp_{d\nu_y} \tau_*(a_{\nu *} f^*[N]) \cap \tau_* df^*[CC(\alpha)]$$

$$= \sharp df_*(a_{\nu *}[M] \cap df^*[CC(\alpha)])$$

$$= \sharp df_* a_{\nu *}[M] \cap [CC(\alpha)]$$

$$= \sharp d(\nu \circ f)[M] \cap [CC(\alpha)] ,$$

where we have used twice the projection formula and $\sharp$ stands for the intersection number. By the Intersection Formula §4.2 combined with the conic structure (as in Example 2.5) the last expression is equal to

$$\chi(\{\nu \circ f = 0\}; \alpha) = f_*(\alpha)(y) .$$

This completes the proof of the functoriality of CC. The functoriality of $c^*(T^*M) \cap s_*(\cdot)$ under this proper push-forward of Lagrangian cycles is much simpler. The main point then is the fact, that $c^*(T^*M) \cap s_*(\xi)$ is just the sum of all components of $[\hat{\xi}] \in H_*(P(T^*M))$ in the decomposition of $H_*(P(T^*M))$ as in the projective bundle theorem 1.2.1 (compare [Al2, lem.4.2, 4.3]).

5 Verdier-Riemann-Roch theorem and Milnor classes

We will assume, that the considered spaces are algebraic or complex analytic.

5.1 Multiplicativity

Chern-MacPherson classes were invented as the unique classes which are functorial with respect to proper push-forwards. It turns out that they behave well with respect to the exterior products. By [Kw] we have

$$c_*(\alpha \times \beta) = c_*(\alpha) \times c_*(\beta)\,.$$

5.2 Verdier-Riemann-Roch theorem

Let $i : X \subset Z$ be a regular embedding and $p : Z \longrightarrow Y$ be a smooth submersion. All spaces are allowed to have singularities. The normal cone of X in Z is the normal bundle $C_X Z = N_X Z$. Let T_p be the tangent bundle of the fibers.

Question. Does the following diagram commute?

$$\begin{array}{ccc} CF(Y) & \xrightarrow{c_*} & \mathcal{H}_*(Y) \\ \downarrow{\scriptstyle p^*} & & \downarrow{\scriptstyle p^*\cap c^*(T_p)} \\ CF(Z) & \xrightarrow{c_*} & \mathcal{H}_*(Z) \\ \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*\cap c^*(N_X Z)^{-1}} \\ CF(X) & \xrightarrow{c_*} & \mathcal{H}_*(X) \end{array}$$

According to Yokura [Yo] the answer is as follows:

Claim 5.1 *The upper square commutes.*

The lower square does not commute in general. The proof of the Yokura result is the following: we have to show that

$$c_*(p^*\alpha) = c^*(T_p) \cap p^* c_*(\alpha)\,.$$

We can assume that $\alpha = q_* 1_M$ with smooth M (we use a resolution of singularities). The map p is covered by a map p'

$$\begin{array}{ccccc} Z \times_Y M & = & M' & \xrightarrow{p'} & M \\ & & \downarrow{\scriptstyle q'} & & \downarrow{\scriptstyle q} \\ & & Z & \xrightarrow{p} & Y \end{array}\,.$$

Note that M' is smooth and $c^*(TM') = c^*(T_{p'}) \cup p'^* c^*(TM)$. Then

$$\begin{aligned} c_*(1_{M'}) &= c^*(TM') \cap [M'] \\ &= (c^*(T_{p'}) \cup p'^* c^*(TM)) \cap [M'] \\ &= c^*(T_{p'}) \cap (p'^* c^*(TM) \cap [M']) \\ &= c^*(T_{p'}) \cap (p'^* c_*(1_M)) \end{aligned}$$

Now we apply q'_* to both sides. The left hand side can be rewritten as

$$q'_* c_*(1_{M'}) = c_*(q'_* 1_{M'}) = c_*(p^*\alpha)\,.$$

The right hand side is equal to

$$q'_*(c^*(q'^* T_p) \cap (p'^* c_*(1_M))) = q'_*(q'^* c^*(T_p) \cap (p'^* c_*(1_M)))$$

and by the projection formula it is equal to

$$c^*(T_p) \cap (q'_* p'^* c_*(1_M)) = c^*(T_p) \cap (p^* q_* c_*(1_M)) =$$
$$= c^*(T_p) \cap (p^* c_* q_*(1_M)) = c^*(T_p) \cap (p^* c_* \alpha)\,.$$

To see that the lower diagram does not commute in general, consider a regular embedding of X into a smooth $Z = M$. On one hand

$$c_*(i^*(1_M)) = c_*(1_X)$$

is the Chern-MacPherson class. On the other hand

$$c^*(N_X M)^{-1} \cap i^*(c_*(1_M)) = c^*(N_X M)^{-1} \cap i^*(c^*(TM) \cap [M]) =$$
$$= (c^*(N_X M)^{-1} \cup i^*(c^*(TM))) \cap [X] = c_*^{FJ}(X)$$

is the Fulton-Johnson class, which in general differs from the MacPherson class.

5.3 Definition of Milnor class

The difference between MacPherson class and Fulton-Johnson class is usually called Milnor class. In our approach we do not have to assume that $Z = M$ is smooth.

Definition 5.2 For a regular embedding $X \subset Z$, and a constructible function $\alpha \in CF(Z)$, the difference

$$\mathcal{M}(X \subset Z; \alpha) = c^*(N_X Z)^{-1} \cap i^* c_*(\alpha) - c_*(i^*\alpha) \in \mathcal{H}_*(X)$$

is called *Milnor class of the pair $X \subset Z$ relative* to α.

Let k be the zero–section and π be the projection in $N_X Z$. By 5.1 for π and $i^*\alpha$

$$c_*(\pi^* i^* \alpha) = c^*(T_\pi) \cap \pi^* c_*(i^*\alpha)\,.$$

We apply k^* to both sides:

$$k^* c_*(\pi^* i^* \alpha) = k^*(c^*(T_\pi) \cap \pi^* c_*(i^*\alpha)) =$$
$$= c^*(N_X Z) \cap k^* \pi^* c_*(i^*\alpha) = c^*(N_X Z) \cap c_*(i^*\alpha)\,.$$

Then $c_*(i^*(\alpha))$ can be rewritten as

$$c^*(N_X Z)^{-1} \cap k^*(c_*(\pi^* i^*(\alpha)))\,.$$

Thus

$$\mathcal{M}(X \subset Z; \alpha) = c^*(N_X Z)^{-1} \cap \Big(i^* c_*(\alpha) - k^*(c_*(\pi^* i^*(\alpha)))\Big)\,.$$

5.4 Milnor class can be computed from vanishing cycles

Note that $i^* = k^* \circ sp$, where $sp : H_*^{BM}(Z) \to H_*^{BM}(C_X Z)$ is the homology specialization map (cf. [Ve1] and [Fl, Ch.5]). Hence

$$i^* c_*(\alpha) = k^* sp\, c_*(\alpha)\,.$$

We state without a proof the following crucial property:

Theorem 5.3 *Let $X \subset Z$ be a closed embedding and let $\alpha \in CF(Z)$. Then*

$$sp(c_*(\alpha)) = c_*(sp(\alpha)) \in H_*^{BM}(C_X Z)\,.$$

(For a sketch of the proof and discussion of a general context of this theorem, due essentially to Verdier [Ve1], we refer to [Sch2, Thm 1.1].)

According to 5.3 we rewrite the preceding formula

$$i^* c_*(\alpha) = k^* c_* sp(\alpha)\,.$$

Corollary 5.4 *The Milnor class can be expressed as*

$$\mathcal{M}(X \subset Z; \alpha) = c^*(N_X Z)^{-1} \cap k^* c_*(\Phi_i(\alpha))\,,$$

where

$$\Phi_i : CF(Z) \to CF_{mon}(C_X Z)\,,$$
$$\Phi_i(\alpha) = sp(\alpha) - \pi^* \circ i^*(\alpha) \in CF_{mon}(C_X Z)\,.$$

Note that $k^* \Phi_i(\alpha) = 0$.

We will now show that $\Phi_i(\alpha)$ is a generalization of the vanishing cycle operation ϕ_f as defined in 2.4.8 [3]. Assume that $X \subset Z$ is a regular embedding of codimension 1. Then X is locally described by a function $f : Z \to A^1$ i.e. $X = f^{-1}(0)$. Let us consider the deformation space M and the map $g : M \to A^1$ such that $sp(\alpha) = \psi_g(\tilde{\pi}^* \alpha)$, see 2.4.9. The function f determines a section s of $\tilde{\pi}$. The image of s does not intersect the zero-section. We have $gs = f$.

$$\begin{array}{ccccc} & & A^1 & & \\ & {\scriptstyle g}\nearrow & & \nwarrow{\scriptstyle pr_2} & \\ M & & \xrightarrow{\ bl\ } & & Z \times A^1\,. \\ & {\scriptstyle \tilde{\pi}}\searrow\nwarrow{\scriptstyle s} & & \nearrow{\scriptstyle 1\times f} & \\ & & Z & & \end{array}$$

The Milnor fiber of g at points away from the zero-section of $N_X Z$ is the product of the Milnor fiber of f and the disk. Thus

$$s^* sp(\alpha) = s^* \psi_g(\tilde{\pi}^*(\alpha)) = \psi_{gs}(\alpha) = \psi_f(\alpha)\,,$$

[3]This paragraph is added by PP-AW.

and

$$s^*(\Phi_i(\alpha)) = \psi_f(\alpha) - i^*(\alpha) = \phi_f(\alpha)\,.$$

In particular, the formula from Corollary 5.4 gives a generalization of Parusiński-Pragacz's formula, [PaPr2], for the Milnor class of a hypersurface, to the higher codimension case. In fact, in [PaPr2], the formula is stated equivalently in terms of a Whitney stratification of the hypersurface, i.e. in the form conjectured by Yokura, generalizing an earlier work of Parusiński-Pragacz [PaPr1] on the Euler characteristic of a singular hypersurface. The method of [PaPr2] relies on the computation of suitable characteristic cycles [LeMe], [BMM], whereas our approach is based on Verdier's specialization [Ve1, Ve2], [Sch2].

5.5 Another view on Milnor class

Assume, as before, that $\operatorname{codim} X = 1$. Then monodromic functions in $N_X Z \setminus X$ are determined by functions on X. Any section of $N_X Z \setminus X$ gives us a local identification

$$CF(X) \simeq CF_{mon}(C_X Z \setminus X)\,.$$

In this case the vanishing cycle transformation passes to a map

$$\mu_i : CF(Z) \to CF(X)\,.$$

It satisfies the formula

$$\Phi_i(\alpha) = \pi^*\mu_i(\alpha) - k_*\mu_i(\alpha)\,.$$

Then

$$\begin{aligned} k^*c_*(\Phi_i(\alpha)) &= k^*c_*(\pi^*\mu_i(\alpha) - k_*(\mu_i(\alpha))) = \\ &= k^*c_*\pi^*\mu_i(\alpha) - k^*c_*k_*\mu_i(\alpha) = \\ &= k^*(c^*(T_\pi) \cap \pi^*c_*\mu_i(\alpha)) - k^*k_*c_*\mu_i(\alpha) = \\ &= c^*(N_X Z) \cap c_*\mu_i(\alpha) - c^1(N_X Z) \cap c_*\mu_i(\alpha) = \\ &= (c^*(N_X Z) - c^1(N_X Z)) \cap c_*(\mu_i(\alpha)) = c_*(\mu_i(\alpha))\,. \end{aligned}$$

Hence

$$\mathcal{M}(X \subset Z; \alpha) = c^*(N_X Z)^{-1} \cap k^*c_*(\Phi_i(\alpha)) = c^*(N_X Z)^{-1} \cap c_*(\mu_i(\alpha))\,.$$

For higher codimension complete intersection there exists a constructible function $\mu_i(\alpha) \in CF(X)$ such, that

$$\Phi_i(\alpha) = \pi^*\mu_i(\alpha) - k_*\mu_i(\alpha) + \text{correction terms.}$$

The correction terms are supported in lower dimensions. Then

$$k^*c_*\Phi_i(\alpha) = (c^*(N_X Z) - c^{\text{top}}(N_X Z)) \cap c_*(\mu_i(\alpha)) + \text{correction terms}$$

and

$$\mathcal{M}(X \subset Z; \alpha) = c^*(N_X Z)^{-1} \cap k^* c_*(\Phi_i(\alpha)) =$$
$$= c^*(N_X Z)^{-1} \cap (c^* N_X Z - c^{top} N_X Z) \cap c_*(\mu_i(\alpha)) + \text{correction terms.}$$

This matter and some related topics are discussed in the two letters of J. Schürmann, reproduced in the following appendix.

6 Appendix: Two letters of J. Schürmann

Letter addressed to Brasselet, Lehmann, Seade, and Suwa, dated December 2001:

Dear colleagues,

I have now looked in more detail at the more complete version of your paper "Milnor-classes of local complete intersections" [BLSS], and I have thought a little bit more about a question of Prof. Suwa.

First I should say that my approach is very difficult for explicit calculations (but it fits better with functorial properties, and in the algebraic context it gives (localized) Milnor-classes in the Chow group). I think a complete comparison of my formula with your corollary 5.13 seems to be very difficult, but I think at least some parts of your formula are related to my approach:

Let me follow the notations of page 18 of [Sch2]: i.e. S_α is a connected component of the singular locus of X, S'_α is an irreducible component of S_α and let me denote the "generic value" of $\Phi_i(1_Y)$ restricted to $N_X Y|S'_\alpha$ by μ'_α. The description

$$\Phi_i(1_Y) = \sum \mu'_\alpha(\pi^* - k_*)(1_{S'_\alpha}) + \beta\,,$$

with β some constructible function which is generically vanishing (and which induces in some sense a correction-term in the calculation of the localized Milnor-class). The sum is over all irreducible components of the singular locus of X, with $\pi : N_X Y \to X$ the projection of the normal bundle and $k : X \to N_X Y$ the zero-section.

So for the constructible function $\mu'_\alpha(\pi^* - k_*)(1_{S'_\alpha})$ one can make the same calculation as on page 8 of [Sch2] (for the codimension one case), and gets by the self-intersection formula the equation

$$A := k^* c_*(\mu'_\alpha(\pi^* - k_*)(1_{S'_\alpha})) = \mu'_\alpha(c^*(N) - c^{top}(N)) \cap c_*(1_{S'_\alpha})\,.$$

So one sees that one gets from each irreducible component the contribution $c^*(N)^{-1} \cap A$ to the (localized) Milnor-class (plus a correction-term coming from β). In particular, for a smooth component S of the singular locus this is just $\mu'_\alpha(c^*(N) - c^{top}(N))\ c^*(N)^{-1}\ c^*(S) \cap [S]$, i.e. the first term of the formula of your corollary 5.13, with the identification of μ'_α as the Milnor-number of a generic transversal slice $X \cap H$!

Here I would expect that one can deduce this calculation of μ'_α from my general approach by some "generic base-change" argument, but the details of this have to be worked out!

To finish this letter I would like to ask you if your calculation of this generic value would also work without the assumption that S is smooth, i.e. at generic points of the smooth part of the singular locus? (so that one gets in this way the top-dimensional "localized Milnor-class" as in the last formula of your corollary 5.13, without the assumption that S is smooth, similarly to the remark on page 18 of [Sch2]).

Best wishes,

Jörg Schürmann

Letter addressed to Aluffi, Brasselet, Kennedy, Pragacz, Suwa, and Weber, dated May 7, 2002:

Dear colleagues,

I have now thought about some questions asked at the conference in Warsaw about my Verdier-Riemann-Roch theorem for Chern (Schwartz-MacPherson) classes.

Enclosed are some remarks and also a question:

(a) Consider a regular embedding $X \to Y$ of spaces and the Verdier specialization for constructible functions $sp : CF(Y) \to CF_{mon}(C_X Y)$. Fix $\alpha \in CF(Y)$, and denote by $sp(\alpha)_{gen} \in CF(X)$ the unique constructible function such that $sp(\alpha) = \pi^*(sp(\alpha)_{gen})$ on a Zariski-dense (conic) open subset of $C_X Y$, with $\pi : C_X Y \to X$ the projection of the normal cone= normal bundle (i.e. write $sp(\alpha)$ as a (locally finite) sum of terms m_i times indicator functions of closed irreducible subcones C_i. Then $sp(\alpha)_{gen}$ is the sum of m_i times the indicator function of $\pi(C_i)$, where we only consider those C_i for which $C_i = \pi^{-1}(\pi(C_i))$).

If the regular embedding is of codimension one, given locally by one equation $g = 0$, then one knows $sp(\alpha)_{gen} = \psi_g(\alpha)$ as in (SP6) of [Sch2] (with ψ_g the nearby cycle functor of Deligne on the level of constructible functions). Here is now the generalization to the higher codimension case: Suppose X is locally given by $f_1 = \ldots = f_n = 0$ for a regular sequence $f_1, \ldots, f_n$. Then one gets:

$$sp(\alpha)_{gen} = \psi_{f_n}(\ldots(\psi_{f_1}(\alpha))\ldots) \quad (1)$$

with the right iterated nearby cycle functor as in [McPa2].

The proof of (1) is by induction similarly to an argument of Verdier on p.204 in Astérisque 36-37 (1976) !

(b) The formula (1) shows in particular, that this iterated nearby cycle functor (on the level of constructible functions) does not (!!) depend on the order of the local tuple $f_1, \ldots, f_n$ (as in the more general context studied by McCrory-Parusiński). So in the case of a regular embedding one can speak of the Euler characteristic of "the Milnor-fiber" of the function $f = (f_1, \ldots, f_n)$ (weighted by α), which for an isolated singularity (of Y and f) reduces to the usual notion! In particular, this gives a proof of my claim on p.18 of [Sch2] that for isolated complete intersection singularities the localized Milnor-class is given by the "usual Milnor-number" (up to signs) as in the paper [Suw] of Suwa!

(c) This (iterated) nearby cycle functor commutes with restriction to transversal slices (with respect to suitable Whitney-stratification) as for example proved in Part V, lemma 3.5 of my book [Sch1] In particular, I get for the generic value μ_α on p.18 of [Sch2] the local description: (Euler-characteristic of the Milnor-fiber of f restricted to a suitable normal slice) -1 . And this implies my expected formula for the top localized Milnor-class. In particular, my top localized Milnor-class is the same as the one of [BLSS] !

(d) There is one important additional case, where one gets more information on $sp(\alpha)$, and not only on its generic value (which I have not mentioned in Warsaw). This comes from the fact that sp for constructible functions commutes with exterior products (!!). So one can for example describe completely $sp(1_Y)$ for a cartesian product of hypersurfaces (and then my approach to Milnor-classes implies the formulas of Ohmoto-Yokura [OhYo] for this special case !

Consider two closed embeddings $X \to Y$ and $X' \to Y'$. Then one has $C_X Y \times C_{X'} Y' = C_{X \times X'}(Y \times Y')$ and

$$sp(\alpha) \times sp(\beta) = sp(\alpha \times \beta)! \tag{2}$$

For the proof of (2) I use that ψ_f commutes with fiber-products (which follows easily from corresponding Milnor-fibrations and the Künneth-formula) , and the following property. Let $M(X, Y), M(X', Y')$ and $M(X \times X', Y \times Y')$ be the corresponding deformation spaces to the normal cones. Then I can define a natural morphism:

$$M(X \times X', Y \times Y') \to M(X, Y) \times_{A^1} M(X', Y') \tag{3}$$

of the first deformation space to the fiber-product of the other ones (which is an isomorphism of the underlying sets) such that over $\{0\}$ one gets the natural isomorphism $C_{X \times X'}(Y \times Y') = C_X Y \times C_{X'} Y'$, and over $A^* = A^1 \setminus \{0\}$ the natural isomorphism

$$Y \times Y' \times A^* = (Y \times A^*) \times_{A^*} (Y' \times A^*).$$

This natural map (3) is induced by the ruled join construction of the two blow-up spaces

$$Bl_{X \times \{0\}}(Y \times A^1) \times Bl_{X' \times \{0\}}(Y' \times A^1) \leftarrow \text{Join}$$

and the natural closed embedding

$$Bl_{X \times X' \times \{0\}}(Y \times Y' \times A^1) \to \text{Join}\,.$$

Here I would expect that this map (3) is an isomorphism, but I didn't checked this (since it is not important for my argument).

I would like finally ask you, if you know of any reference for the comparison of the deformation space

$$M(X \times X', Y \times Y')$$

to the fiber product $M(X,Y) \times_{A^1} M(X',Y')$?

I do not know any reference in the literature to something of this type. So if someone of you knows about this, please let me know .

Best wishes,

Jörg

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