Purity at the end

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Abstract

We consider smooth completion of algebraic manifolds. Having some information about its singular completions or about completions of its images we prove purity of cohohomology of the set at infinity. We deduce also some topological properties. The work is based on the study of perverse direct images for algebraic maps.

Key words and phrases. Algebraic varieties, weight filtration, perverse sheaves.

1 Introduction

This paper is inspired by the important paper of De Cataldo and Migliorini [3], where the authors give a proof of Decomposition Theorem of [1, Théorème 6.2.5] using Hodge-theoretic methods. As a side result they obtain some contractibility criteria for subvarieties. We present a generalizations of these criteria. Our proof is based on Purity Theorem of Gabber [4] or stability of pure perverse sheaves with respect to intermediate extension [1, Corollarie 5.3.2]: for a pure perverse sheaf K supported by an open set the intermediate extension $j_{!*}K$ is pure. We use formally the properties of the derived category of sheaves and our argument works equally well in the setup of Weil sheaves (as in [1]) and for mixed Hodge modules of [6].

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We study open smooth algebraic varieties and their smooth completions. Let $X = \overline{U}$ be a completion of U. Our reasoning is localized around a connected component Z of the set $X \setminus U$, which we call an and of U. Knowing that some image $V = f(U)$ of U admits a (possibly singular) completion Y we study the weight filtration in the cohomology of the link of Z.

Let us explain the case when $\overline{f} : X \to Y$ is a resolution of an isolated singularity, $Z \subset X$ is the exceptional locus, $U = X \setminus Z$, $V = Y \setminus \overline{f}(Z)$, $f = \overline{f}_{|U}$. We have an exact sequence

$$\to \quad H^{k-1}(U) \quad \overset{\alpha}{\to} \quad H^k(X,U) \quad \overset{\beta}{\to} \quad H^k(X) \quad \overset{\gamma}{\to} \quad H^{k-1}(U) \quad \to .$$

The following argument is valid for complex varieties, but with some suitable changes[1] can be performed in the category of varieties over a finite characteristic field (see the proof of Theorem 8). We present it to give a topological motivation of further constructions. We replace Y by a conical neighbourhood of the singular point $\{p\} = f(Z)$. Then X, the resolution of Y, is a manifold with boundary, which retracts to the exceptional set Z. The boundary ∂X is homeomorphic to the link of p in Y, denoted by $L_p(Y)$. The open set U retracts to $\partial X \simeq L_p(Y)$. Let $m = \dim X$. By Poincaré duality we have $H^k(X,U) \simeq H_{2m-k}(Z)$. The long exact sequence can be rewritten in the following way:

$$\to \quad H^{k-1}(L_p(Y)) \quad \overset{\alpha}{\to} \quad H_{2m-k}(Z) \quad \overset{\beta}{\to} \quad H^k(Z) \quad \overset{\gamma}{\to} \quad H^k(L_p(Y)) \quad \to .$$

All the cohomology groups appearing in the diagram are equipped with weight filtration. Since Z is complete

- $H^k(X,U)$ is of weight $\geq k$, (the isomorphism with $H_{2m-k}(Z) = \big(H^{2m-k}(Z)\big)^*$ shifts the weight by $2m$),
- $H^k(Z)$ is of weight $\leq k$,

[1] Although the notion of the link $L_W(Y)$ of a subvariety W in Y is ambiguous (and has no meaning for varieties over $\mathbb{F}_q$) it is justified to talk about the cohomology $H^*(L_W(Y);\mathbb{K}) := H^*(W; i^* j_* j^* \mathbb{K}_Y)$, where $j : Y \setminus W \to Y$ and $i : W \to Y$ are inclusions.

Moreover, by purity of the intersection sheaf IC_Y we have

- for $k < m$ the group $H^k(L_p(Y)) \simeq \mathcal{H}^k(IC_Y[-m])_p$ is of weight $\leq k$,
- by duality, for $k \geq m$ the group $H^k(L_p(Y)) \simeq H^{k+1}_{\{p\}}(Y; IC_Y[-m])$ is of weight $\geq k$,

see [2]. It follows that for $k \leq \dim Y$ the map α vanishes and the map β is injective. (Dually, for $k \geq \dim Y$ the map γ vanishes and β is surjective.) The case of isolated singularities was already studied in [5] as a corollary from the decomposition theorem. Additionally, from injectivity of α we obtain that $H^k(X, U)$ is pure of weight k for $k \leq \dim X$. By duality we have

Theorem 1 *For a resolution of an isolated singularity $\overline{f} : X \to Y$ the cohomology of the exceptional set $H^k(Z)$ is pure of weight k for $k \geq \dim X$.*

We would like to emphasize that by [7] the condition that $H^k(Z)$ is pure of weight k depends only on topology of Z.

The Theorem 1 is proved by de Cataldo and Migliorini, as a corollary from their proof of Decomposition Theorem. In fact it is enough to assume that the map f is semismall, [3, Th. 2.1.11]. We generalize the Theorem 1 without any assumption for the map $f : U \to V$. Instead we have a condition for the degree k for which the cohomology group $H^k(Z)$ is pure. The Decomposition Theorem of [1] or [6] is involved in our proof but we do not rely on it, although we have to use equally strong arguments as purity of the intermediate extension of a pure perverse sheaf.

Our argument is formal. We list below the formal properties of weights which we use in the proof. These properties hold equally well for Weil sheave of [1] and for mixed Hodge modules of M. Saito [6]. In the first case we apply ℓ-adic étale cohomology for varieties defined over a finite field $\mathbb{F}_q$ ($\ell \neq char(\mathbb{F}_q)$), that is $H^*(X_0; F_0) := H^*_{et}(X_0 \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q; F)$, where $F_0 \in \mathcal{D}(X_0)$

is a Weil sheaf, see the notation of [1, §5.1]. These cohomology groups are vector spaces over $\overline{\mathbb{Q}}_\ell$, the closure of the field of ℓ-adic numbers. In the second case the cohomology is a vector space over rationals or complex numbers. Depending on the context let $\mathbb{K}$ denote $\overline{\mathbb{Q}}_\ell$, $\mathbb{Q}$ or $\mathbb{C}$.

Notation:

– $\mathcal{D}(X)$ denotes the category of Weil sheaves or the category of mixed Hodge modules. The objects of $\mathcal{D}(X)$ are called sheaves.

– We drop the letter R for the direct image of an object of $\mathcal{D}(X)$, i.e. we write f_*F instead of Rf_*F for $f : X \to Y$.

– On the other hand for the constant sheaf $\mathbb{K}_X$ we write $Rf_*\mathbb{K}_X$ since $\mathbb{K}_X$ is a sheaf in the usual sense.

– For the truncation functor we write $\tau_{<d}$ instead of $\tau_{\leq d-1}$.

We list the formal properties of weights which are used in the proofs.

(0.1) functors $f_!$ and f^* preserve the weight condition $\leq w$,

(0.2) in particular if $F \in \mathcal{D}(X)$ is of weight $\leq w$, then $H^k_c(X; F)$ is of weight $\leq w + k$,

(0.3) functors f_* and $f^!$ preserve the weight condition $\geq w$,

(0.4) in particular if $F \in \mathcal{D}(X)$ is of weight $\geq w$, then $H^k(X; F)$ is of weight $\geq w + k$,

(0.5) Verdier duality switches the weight conditions $\geq w$ and $\leq -w$,

(0.6) Verdier duality exchanges ! with $*$, i.e. $D \circ f^! = f^* \circ D$ and $D \circ f_! = f_* \circ D$,

2 Main Theorem

The set of components $U_\infty = \pi_0(X \setminus U)$ for a normal completion X of a normal algebraic variety U does not depend on the completion. An element of this set is called an end of U. A map of algebraic varieties which is proper induces a map of their ends. A completion of an end $\eta \in U_\infty$ is the component $Z \subset X \setminus U$ corresponding to η. We say that the completion Z of η is smoothing if X is smooth in a neighbourhood of Z. Below we give the exact statement of our main theorem.

Theorem 2 *Suppose we have a proper map of varieties $f : U \to V$ with smooth U of dimension m and let $r = r(f)$ be the defect of semismallness. Let η be an end of U. Suppose that $f(\eta) \in V_\infty$ admits a completion with the variety at the end of dimension d. Then for any smoothing completion Z of η the cohomology $H^k(Z)$ is pure for the degrees $k \geq m+r+d$. Moreover, for that range of degrees the restriction map $H^k(X,U) \to H^k(Z)$ is surjective.*

The notion of the defect of semismallness was invented in [3], we recall it in Definition 3. If f is a fibration then $r(f)$ is the dimension of the fiber.

In the proof we reduce the general situation to the case when $f : U \to V$ extends to $\overline{f} : X \to Y$. Then Theorem 2 follows from Corollary 9 which is the dual version of Theorem 8. There we deal with an arbitrary pure sheaf F on Y instead of the sheaf $R\overline{f}_*\mathbb{K}_X$ and instead of extracting geometric conditions of f we record vanishing range of perverse cohomology ${}^p\mathcal{H}^k(F)$.

3 Defect of semismallness

Assume that the variety U is smooth. The following discussion is to clarify the relation between the decomposition [1, Théorème 6.2.5]

$$Rf_*\mathbb{K}_U = \bigoplus IC_{S_\alpha}(L_\alpha)[d_\alpha]$$

and the geometric properties of the map f. We recall the definition of a number which measures to what extend $Rf_*\mathbb{K}$ is not perverse.

Set

$$V^i = \{y \in V : \dim f^{-1}(y) = i\}\,.$$

Definition 3 [3, Def 4.7.1] The *defect of semismallness* of a map $f : U \to V$ is the integer

$$r(f) = \max_{i:V^i \neq \emptyset} \{2i + \dim V^i - \dim U\}\,.$$

Note that

- $r(f) = 0$ if and only if f is semismall.
- If f is a fibration, then $r(f)$ is equal to the dimension of the fiber.

The defect of semismallness controls the number of perverse derived images which appear in the Decomposition Theorem. To see that we need to recall the perverse t-structure in the derived category $\mathcal{D}(V)$, according to which for a closed smooth variety $W \subset Y$ the sheaf $\mathbb{K}_W[\dim W]$ is perverse. Precisely:

$$F \in {}^p\mathcal{D}(V)^{\leq 0} \quad \text{if } \forall\, s\ \dim(\operatorname{Supp} \mathcal{H}^s F) \leq -s\,.$$

Then

$$F \in {}^p\mathcal{D}(V)^{\leq r} \quad \text{if } \forall\, s\ \dim(\operatorname{Supp} \mathcal{H}^s F) \leq r - s\,.$$

If we apply this condition to $F = Rf_*\mathbb{K}_U[m]$ we obtain

$$Rf_*\mathbb{K}_U[m] \in {}^p\mathcal{D}(V)^{\leq r} \quad \text{if } \forall\, t\ \dim(\operatorname{Supp} R^t f_*\mathbb{K}_U) \leq m + r - t\,. \qquad (*)$$

Note that for V^i introduced before the definition of the defect of semismallness we have

$$\overline{V^i} = \operatorname{Supp} R^{2i} f_*\mathbb{K}_U\,.$$

Hence the condition (*) is satisfied for even $t = 2i$ provided that $\dim V^i \leq m + r - 2i$. For odd $t = 2i - 1$ we have $\operatorname{Supp} R^{2i-1} f_*\mathbb{K}_U \subset \operatorname{Supp} R^{2i} f_*\mathbb{K}_U$, so (*) follows from the condition for $t = 2i$. Therefore the condition (*) is satisfied for all t provided that the map has the defect of semismallnes at most to r.

We conclude that for any proper map $f : U \to V$ with $r(f) = r$ we have a decomposition

$$Rf_*\mathbb{K}_U[m] \simeq \bigoplus_{s=-r}^{r} {}^p\mathcal{H}^s(Rf_*\mathbb{K}_U[m])[-s] \,.$$

Note that the perverse cohomology for degrees smaller than $-r$ vanish since $Rf_*\mathbb{K}_U[m]$ is self-dual (i.e. $D(Rf_*\mathbb{K}_U[m]) \simeq Rf_*\mathbb{K}_U[m]$).

4 Proof of the main theorem

Let Y be an algebraic variety. Let $W \subset Y$ be a closed subvariety of dimension d and let $V = Y \setminus W$. Denote by j the inclusion $V \hookrightarrow Y$.

Lemma 4 *Let P be a perverse sheaf on $Y \setminus W$, which is of weight $\leq w$, then for $y \in W$ the cohomology stalks $\mathcal{H}^k((j_*P)_y)$ are of weight $\leq k + w$ for $k < -d$. In another words $\tau_{<-d}\, j_*P$ is of weight $\leq w$.*

Proof. The natural map

$$\tau_{<-d}\, j_{!*}P \to \tau_{<-d}\, j_*P$$

is an isomorphism by construction [1, Proposition 2.1.17]. The sheaf $j_{!*}P$ is of weight $\leq w$ by the stability of weight with respect to the intermediate extension [1, Corollarie 5.3.2]. $\square$

We fix a natural number r and consider the sheaves which have the perverse cohomology bounded below by $-r$. We obtain the corollary:

Corollary 5 *Let $F \in {}^p\mathcal{D}^{\geq -r}(V)$, be a sheaf which is of weight $\leq w$. Then for $y \in W$ the cohomology stalks $\mathcal{H}^k((j_*F)_y)$ are of weight $\leq k + w$ for $k < -d - r$. In another words $\tau_{<-d-r}\, j_*P$ is of weight $\leq w$.*

Proof. If F is of weight $\leq w$ then its preverse cohomology ${}^p\mathcal{H}^k(F)$ is of weight $\leq w+k$. In the framework of Weil sheaves this is [1, Théorème 5.4.1]. In Saito construction this is the definition [6, 4.5.1]. By usual spectral sequence argument we deduce the claim from Lemma 4. □

Remark 6 We will apply Corollary 5 for pure sheaves F. In that case we do not have to use spectral sequences since F is isomorphic to the sum of its perverse cohomology, i.e.

$$F = \bigoplus_{s\geq r} {}^p\mathcal{H}^s(F)[-s]\,,$$

by [1, Théorème 5.4.5] or by [6, formula 4.5.4].

Let $i : W \hookrightarrow Y$ be the inclusion. Suppose that W is complete.

Lemma 7 *Let $F \in {}^p\mathcal{D}^{\geq -r}(V)$ be a sheaf on V, which is of weight $\leq w$. Then the cohomology $H^k(W; i^*j_*F)$ is of weight $\leq w+k$ for $k < -d-r$.*

Proof. The map $H^k(W; i^*\tau_{<-d-r}\, j_*F) \to H^k(W; i^*j_*F)$ is an isomorphism for $k < -d-r$. The conclusion follows from (0.1), (0.2) and Corollary 5 . □

Here is our key statement:

Theorem 8 *Let $G \in \mathcal{D}(Y)$ such that $j^*G \in {}^p\mathcal{D}^{\geq -r}(Y)$. Assume that W is complete. If G is pure of weight w, then $H^k(W; i^!G)$ is pure of weight $w+k$ for $k \leq -d-r$. Moreover the natural map $H^k(W; i^!G) \to H^k(W; i^*G)$ is injective.*

Proof. We have a distinguished triangle

$$\begin{array}{ccc} i_*i^!G & \longrightarrow & G \\ {\scriptstyle [+1]}\nwarrow & & \swarrow \\ & j_*j^*G. & \end{array}$$

Let us restrict this triangle to W. We have $i^*i_*i^!G = i^!G$ and we obtain the the triangle

$$\begin{array}{ccc} i^!G & \longrightarrow & i^*G \\ {\scriptstyle [+1]}\nwarrow & & \swarrow \\ & i^*j_*j^*G. & \end{array}$$

Consider the associated sequence of cohomology

$$H^{k-1}(W; i^* j_* j^* G) \stackrel{\alpha}{\to} H^k(W; i^! G) \stackrel{\beta}{\to} H^k(W; i^* G) \stackrel{\gamma}{\to} H^k(W; i^* j_* j^* G)$$

(compare the exact sequence from the Introduction). By (0.3) the sheaf $i^! G$ is of weight $\geq w$. By (0.4) the cohomology group $H^k(W; i^! G)$ is of weight $\geq w + k$. On the other hand $i^* G$ is of weight $\leq w$ by (0.1) and since W is complete the cohomology group $H^k(W; i^* G) = H^k_c(W; i^* G)$ is of weight $\leq w + k$. By Lemma 7 applied to $F = j^* G$ the cohomology $H^{k-1}(W; i^* j_* j^* G)$ is of weight $\leq w + k - 1$ for $k - 1 \leq -d - r$. It follows that the map α is trivial. Therefore the map β is injective and the group $H^k(W; i^! G)$ is pure. □

Corollary 9 *Suppose $G \in \mathcal{D}(Y)$ is pure of weight w and $j^* G \in {}^p\mathcal{D}^{\leq r}(Y)$. Then the cohomology $H^k(W; i^* G)$ is pure for $k > d + r$ and the map γ : $H^k(W; i^! G) \to H^k(W; i^* G)$ is surjective.*

Proof. We apply the Theorem 8 for the dual sheaf DG. By the property (0.5) of Verdier duality the assumptions Theorem 8 are satisfied. Hence by (0.6) and Theorem 8

$$H^k(W; i^! DG) = H^{-k}(W; i^* DG)^*, \quad H^k(W; i^* DG) = H^{-k}(W; i^! DG)^*$$

is pure for $k \geq d + r$. □

We specialize Theorem 8 to the following situation. Let $\overline{f} : X \to Y$ be a proper map and let $W \subset Y$ be a complete subvariety of dimension d. Denote by Z the inverse image $f^{-1}(W)$:

$$\begin{array}{ccccc} U & \stackrel{j'}{\hookrightarrow} & X & \stackrel{i'}{\hookleftarrow} & Z \\ \downarrow^{f} & & \downarrow^{\overline{f}} & & \downarrow^{f^{\infty}} \\ V & \underset{j}{\hookrightarrow} & Y & \underset{i}{\hookleftarrow} & W \end{array} .$$

We assume that U is smooth. Let $r = r(f)$ be the defect of semismallness of $f = \overline{f}_{|U}$. The sheaf $G = \overline{f}_* IC_X$ is pure of weight $m = \dim X$ and it satisfies the assumption of the Theorem 8 and Corollary 9. We have $f_*^\infty i'^! = i^! \overline{f}_*$ and $f_*^\infty i'^* = i^* \overline{f}_*$. We obtain the following:

Theorem 10 *With the notation as above we have*

1. $H^k(Z; i^! IC_X)$ *is pure of weight* $m + k$ *for* $k \leq -d - r$,
2. $H^k(Z; i^* IC_X)$ *is pure of weight* $m + k$ *for* $k \geq d + r$,
3. *the map* $\beta : H^k(Z; i^! IC_X) \to H^k(Z; i^* IC_X)$ *is injective for* $k \leq -d - r$ *and surjective for* $k \geq d + r$.

When X is smooth of dimension m we translate the conclusion of Theorem 10 to the cohomology with coefficients in the constant sheaf $\mathbb{K}_Z$. We obtain:

1. $H^k_Z(X; \mathbb{K}) = H^k(X, U; \mathbb{K})$ is pure of weight k for $k \leq m - d - r$,
2. $H^k(Z; \mathbb{K})$ is pure of weight k for $k \geq m + d + r$,
3. the map $\beta : H^k(X, U; \mathbb{K}) \to H^k(Z; \mathbb{K})$ is injective for $k \leq m - d - r$ and surjective for $k \geq m + d + r$.

We have obtained the conclusion of Theorem 2 except that we have to get rid of the assumption that the map $f : U \to V$ extends to $\overline{f} : X \to Y$. To this end consider the variety

$$\widehat{X} = \overline{graph(f)} \subset X \times Y\,.$$

We apply Theorem 10 for $\widehat{f} = pr_Y : \widehat{X} \to Y$ and deduce purity of $H^*(\widehat{Z}; i^! IC_{\widehat{X}})$ Let $g = pr_X : \widehat{X} \to X$ be the projection. The composition of the natural maps

$$\mathbb{K}_X[m] \to g_* \mathbb{K}_{\widehat{X}}[m] \to g_* IC_{\widehat{X}} \to g_* D\mathbb{K}_{\widehat{X}}[m] = Dg_* \mathbb{K}_{\widehat{X}}[m] \to D\mathbb{K}_X[m] = \mathbb{K}_X[m]$$

is the identity. Hence $\mathbb{K}_X[m]$ is a direct summand of $g_*IC_{\widetilde{X}}$ in a natural way. Therefore $H^k(X,U;\mathbb{K})$ is pure of weight k for $k \leq m-d-r$. The purity of $H^k(Z;\mathbb{K})$ follows by duality. Also triviality of the map from $\alpha : H^{k-1}(Z; i^*j_*j^*\mathbb{K}_X[m]) \to H^k(Z; i^!\mathbb{K}_X[m])$ follows by comparison with the covering maps for $\widehat{X}$. We deduce injectivity of the map $\beta : H^k(X,U;\mathbb{K}) \to H^k(Z;\mathbb{K})$ for $k \leq m-d-r$. The surjectivity for the complementary degrees follows by duality. This completes the proof of the Theorem 2.

5 Fibration case

It is interesting to analyze the case when f is a fibration. If the fiber is of dimension r, then the defect of semismallness is equal to r. Set $n = \dim V = m-r$. Then V is a smooth variety of dimension n. The derived direct image decomposes

$$Rf_*\mathbb{K}_U = \bigoplus_{s=0}^{2r} R^s f_*\mathbb{K}_U[-s]\,.$$

We have a decomposition of the cohomology of link of Z:

$$H^k(L_Z(X)) = \bigoplus_{s=0}^{2r} H^{k-s}(L_W(Y); R^s f_*\mathbb{K}_U)\,.$$

By [2] the cohomolgy of the link $H^k(L_W(Y))$ is of weight $\leq k$ for $k < m-d$. Lemma 7 generalized this statement to the case of twisted coefficients (note that $R^s f_*\mathbb{K}_U$ is pure of weight s). Again we see that $H^k(L_Z(X))$ is of weight $\leq k$ for $k < m-d = n-r-d$.

6 Corollaries

We derive some easy corollaries. Fist of all the restriction map $H^k(X,U) \to H^k(Z)$ factors through $H^k(X)$, therefore we have:

Corollary 11 *With the notation of Theorem 2 for $k \geq m+r+d$ the restriction map $H^k(X) \to H^k(Z)$ is surjective.*

In particular when $f : U \simeq V$ we obtain an easy criterion for contractibility:

Corollary 12 *Let Z be a complete subset of an algebraic manifold X. Suppose that $rk\, H^k(X) < rk\, H^k(Z)$ then there does not exist an algebraic variety Y with a subvariety $W \subset Y$ such that $Y \setminus W \simeq X \setminus Z$ and $\dim W \le k - m$.*

When X is complete we obtain an exact sequences of pure cohomology groups:

Corollary 13 *Suppose that X is complete (and smooth as before). With the notation of Theorem 2 for $k > m + r + d$ we have an exact sequences of pure cohomology groups of weight k*

$$0 \to H^k(X, Z) \to H^k(X) \to H^k(Z) \to 0\,.$$

Dually, for $k < m - r - d$ we have an exact sequence of pure cohomology groups of weight k

$$0 \to H^k(X, U) \to H^k(X) \to H^k(U) \to 0\,.$$

Information about the weight structure of $H^*(Z)$ can exclude some of contractions regardless of the ambient space X:

Example 14 Let $C = \mathbf{P}^1/\{0, \infty\}$ be the nodal curve. Then $H^1(C) = \mathbb{K}$ is of weight 0. Let M be a smooth variety of dimension k. Then $H^{2k+1}(M \times C)$ is not pure. Therefore $Z = M \times C$ cannot be shrunk to a point in a variety X of the dimension $n \le 2k + 1$. (E.g. $k = 1$, $n = 3$.) Another example of this type is given in [3, Remark 2.1.13].

Note that the condition that $H^k(Z)$ is pure depends only on the topology of Z. According to [7] the bottom part of the weight filtration $W_{k-1}H^k(Z)$ is equal to the kernel of the natural map to intersection cohomology$H^k(Z) \to IH^k(Z)$, and both invariants depend only on topology. The assumptions of Theorem 2 imply that $H^k(Z) \to IH^k(Z)$ is injective for $k \ge m + r + d$.

Remark 15 The case when f is an isomorphism, $d = 0$, i.e. the case of an isolated singularity was studied by many authors (e.g. [5]). Let Y be a conical neighbourhood of an isolated singularity and X be its resolution, Z the exceptional set (as in the introduction). The middle part of the exact sequence of the pair $(X, \partial X)$ has the form:

$$\to \quad H^{n-1}(\partial X) \quad \stackrel{0}{\to} \quad H^n(X, \partial X) \quad \stackrel{\simeq}{\to} \quad H^n(X) \quad \stackrel{0}{\to} \quad H^n(\partial X) \quad \to .$$

The isomorphism in the center can be translated to the nondegenerate intersection form on $H^n(X, U)$. For the degrees $k < n$ the map $H^k(X) \to H^k(\partial X)$ is a surjection. The kernel is isomorphic to $H^k(X, U)$. Note that since this kernel is pure we obtain that $H^k(\partial X)$ is pure if and only if $H^k(Z)$ is pure. We generalize this remark.

Corollary 16 *With the assumption of Theorem 2 for $k < m - r - d$ the cohomology of the link $H^k(L_Z(X))$ is pure of weight k if and only if $H^k(Z)$ is pure of weight k.*

Remark 17 When X is complete and when there exists a map $\overline{f} : X \to Y$ extending f then the statement of Theorem 2 can be obtained directly applying Decomposition Theorem. We have

$$R\overline{f}_* \mathbb{K}_X[m] \simeq \bigoplus_{s=-r}^{r} {}^p\mathcal{H}^s(\overline{f}_* \mathbb{K}_X[m])[-s] \oplus F \,,$$

where F is a sheaf supported by W. In another words

$$R\overline{f}_* \mathbb{K}_X \simeq A \oplus B \,,$$

where $A \in {}^p\mathcal{D}^{\leq r+m}(Y)$ and $B = F[-m]$ is supported by W. Counting the dimensions of the supports of $\mathcal{H}^k(A)$ we find that $H^k(W; A) = 0$ for $k \geq m + r + d$ and

$$H^k(Z) \simeq H^k(W; (\overline{f}_* \mathbb{K}_X)_{|W}) \simeq H^k(W; B_{|W}) \simeq H^k(Y; B) \,.$$

But the last group is a direct summand in $H^k(X) = H^k(Y; R\overline{f}_* \mathbb{K}_X)$. Hence it is pure. By the exactly the same argument one proves Corollary 9, but we have chosen the way which allows to trace directly how the weights of link cohomology is involved.

References

[1] Beilinson, A. A.; Bernstein, J.; Deligne, P.: *Faisceaux pervers.* Analysis and topology on singular spaces, I (Luminy, 1981), 5-171, Astérisque, 100, Soc. Math. France, Paris, 1982.

[2] Durfee, A. H.; Saito, M.: *Mixed Hodge structures on the intersection cohomology of links,* Compos. Math. 76 (1990), no.1/2, 49-67.

[3] de Cataldo, M. A.; Migliorini, L.: *The Hodge theory of algebraic maps,* Ann. Sci. Éc. Norm. Supér. (4) 38 (2005), no. 5, 693-750.

[4] Gabber, O.: Preprint IHES 1980

[5] Goresky, M.; MacPherson, R.: *On the topology of complex algebraic maps.* Algebraic geometry (La Rábida, 1981), 119-129, Lecture Notes in Math., 961, Springer, Berlin, 1982.

[6] Saito, M.: *Hodge Hodge Modules*, Publ. Res. Inst. Math. Sci. 26 (1990), no. 2, 221–333.

[7] Weber, A.: *Pure homology of algebraic varieties.* Topology Vol. 43, no 3, (2004) p. 635-644.