A MORPHISM OF INTERSECTION HOMOLOGY AND HARD LEFSCHETZ

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Abstract. We consider a possibility of the existence of intersection homology morphism, which would be associated to a map of analytic varieties. We assume that the map is an inclusion of codimension one. Then the existence of a morphism follows from Saito's decomposition theorem. For varieties with conical singularities we show, that the existence of intersection homology morphism is exactly equivalent to the validity of Hard Lefschetz Theorem for links. For varieties with arbitrary analytic singularities we extract a remarkable property, which we call Local Hard Lefschetz.

0. Introduction

Let Y be a complex algebraic variety of pure dimension. Any algebraic subvariety X of dimension i defines a class in the homology group with closed supports:

$$[X] \in H^{\mathrm{cld}}_{2i}(Y;\mathbb{Z})\,.$$

It was conjectured in [Bry], §5 Problèmes ouverts, that any such class lifts to intersection homology of Y. There are examples that it is not the case for integer coefficients. For rational coefficients the answer is positive. Barthel et al. ([BBFGK]) proved that for any morphism of algebraic varieties (not only an inclusion) there exists an associated homomorphism of rational intersection homology, however this morphism is not unique. Unfortunately the methods of the existing proofs are extremely advanced. The key argument is to reduce to a finite characteristic and then to apply the results of [BBD]. Another proof is made by means of resolution of singularities, see [We]. It relies on the decomposition theorem, which again follows from [BBD].

Any inclusion of algebraic varieties may be factorized by inclusions of codimension one. For general *analytic* varieties we cannot make this reduction, but we will assume that we are given a pair of analytic varieties with $\mathrm{codim}(X\subset Y)=1$. We will only consider intersection homology with rational coefficients. This paper has three purposes. First we want to show how to deduce the existence of intersection homology morphisms from the Hard Lefschetz Theorem provided that singularities are conical, see Definition 1.1 and Theorem 1.2. Our method is related to the earlier paper of Brasselet, Fieseler and Kaup [BFK]. Later, we give a proof for general

1991 *Mathematics Subject Classification.* Primary 14S32, 14F17; Secondary 32S60, 32S50.

Key words and phrases. intersection homology, morphism, conical singularity.

Partially supported by KBN 2 P03A 01113 grant

analytic singularities ($\operatorname{codim}(X \subset Y) = 1$), which is based on the decomposition theorem proved by M. Saito for projective morphisms.

The second purpose is the following : Proposition 2.2 asserts that an intersection homology morphism extends to a stratum if and only if the map of intersection homology of its links vanishes in the middle dimension. This is always the case for analytic varieties. The vanishing property are stated in Corollaries 5.1–2. For isolated singularities we have:

Proposition 0.1. *Let $X \subset Y$ be a pair of analytic varieties with $\dim X = n$ and $\dim Y = n + 1$. Suppose that x is an isolated singular point of X and Y. Let $\ell_X \subset \ell_Y$ be its links in X and Y. Then the maps induced by the inclusion of links $\iota_* : H_n(\ell_X) \to H_n(\ell_Y)$ vanishes.*

This assertion can also be derived from the mixed Hodge structure on the links. For nonisolated singularities, it holds when one replaces homology by intersection homology. Moreover, this property of links implies the Hard Lefschetz Theorem as shown in Proposition 5.4. We are tempted to call this property "local Hard Lefschetz". In the algebraic case it follows directly from Lemme clef of [BBFGK], 3.3.

Finally we want to show a new method of constructing elements in intersection homology. Any sequence of subvarieties

$$X = X^0 \supset X^1 \supset \cdots \supset X^k$$

with $\operatorname{codim}(X^i \subset X) = i$ such that no component of X^i is contained in the singularities of X^{i-1} determines a sequence of classes in intersection homology. This method may be applied to lift Chern classes to intersection homology as it is done in [BW].

Let us comment the argument for conical singularities. Our proof of the existence of intersection homology morphisms is dual to the one in [BBFGK]. We expose the geometry which is hidden behind the operation on sheaves. We also state a result about uniqueness. The proof is by induction with respect to strata. The inductive step has two parts. The first: Propositions 2.1 and 2.2 follows from standard operation on sheaves. It remains true for any real stratified space with even dimensional strata. The geometry is explained in Remark 2.5. The mysterious part is Proposition 3.1 which makes sense only for conical singularities. Our approach is a step towards a full geometric proof, but such a proof cannot exists unless there is a geometric proof of the Hard Lefschetz.

I would like to thank Professor Gottfried Barthel for his patience, valuable comments and various improvements of this paper.

1. Morphism of Intersection Homology

Let (Y, X) be a pair of complex analytic varieties with $\operatorname{codim}(X \subset Y) = 1$. Let $\mathcal{IC}^\bullet_Y$ be the intersection homology sheaf with rational coefficients (which is an object of the derived category of sheaves on Y). We follow the convention according to which $\mathcal{IC}^\bullet_Y$ restricted to the nonsingular part of Y is concentrated in the dimension zero. The coefficients are rationals. Then intersection homology groups with closed supports (for the middle perversity $\mathfrak{m}$, which is always tacitly understood unless otherwise stated) are hypercohomology groups [GM1]:

$$IH^{\mathrm{cld}}_i(Y) = \mathbb{H}^{2\dim Y - i}(Y; \mathcal{IC}^\bullet_Y)\,.$$

Let $\mathcal{IC}^\bullet_X$ be the intersection homology sheaf of X considered as a sheaf on Y supported by X. Then

$$IH^{\rm cld}_i(X) = \mathbb{H}^{2\dim X - i}(Y; \mathcal{IC}^\bullet_X)\,.$$

Analogously let $\mathbb{D}^\bullet_Y = C^{\rm cld}_{2\dim Y - \bullet}(Y)$ and $\mathbb{D}^\bullet_X = C^{\rm cld}_{2\dim X - \bullet}(X)$ be the sheaves of Borel-Moore chains. The homology map is induced by a sheaf morphism:

$${\rm incl}_\# : \mathbb{D}^\bullet_X[-2] \to \mathbb{D}^\bullet_Y\,,$$

where

$$(\mathbb{D}^\bullet_X[-2])_i = (\mathbb{D}^\bullet_X)_{i-2} = C^{\rm cld}_{2\dim X - i+2}(X) = C^{\rm cld}_{2\dim Y - i}(X)\,.$$

We are looking for a morphism

$$\alpha_{X,Y} : \mathcal{IC}^\bullet_X[-2] \to \mathcal{IC}^\bullet_Y\,,$$

which is *compatible with homology*, i.e. makes the following diagram commutative:

$$\begin{array}{ccc} \mathcal{IC}^\bullet_X[-2] & \xrightarrow{\alpha_{X,Y}} & \mathcal{IC}^\bullet_Y \\ \downarrow & & \downarrow \\ \mathbb{D}^\bullet_X[-2] & \xrightarrow{{\rm incl}_\#} & \mathbb{D}^\bullet_Y\,. \end{array}$$

The vertical arrows in the diagram are the canonical morphisms from intersection homology to homology. Assume that no component of X is contained in the singularities of Y. Let $\mathcal{I}_\mathfrak{p}\mathcal{C}^\bullet_Y$ be the intersection homology sheaf for the logarithmic perversity $\mathfrak{p}$:

$$\mathfrak{p}(2c) = \mathfrak{m}(2c) + 1 = c \qquad \text{for} \quad c > 0\,.$$

Suppose that $\mathcal{S}$ is a Whitney stratification of the pair (Y, X). Let $Y_{\rm reg}$ be the nonsingular part of Y. Construct another stratification $\mathcal{S}'$ of Y for which $Y_{\rm reg}$ is the biggest stratum and $Y \setminus Y_{\rm reg}$ is stratified by $\mathcal{S}$. If a geometric chain in X is allowable with respect to $\mathfrak{m}$ and to the stratification $\mathcal{S} \cap X$, then it is allowable in Y with respect to $\mathfrak{p}$ and $\mathcal{S}'$. Thus the inclusion of geometric chains $C^{\rm cld}_\bullet(X) \subset C^{\rm cld}_\bullet(Y)$ gives us a morphism of sheaves

$${\rm Incl}_\# : \mathcal{IC}^\bullet_X[-2] \to \mathcal{I}_\mathfrak{p}\mathcal{C}^\bullet_Y\,.$$

We will say that a morphism $\alpha_{X,Y}$ is *geometric* if the following triangle commutes:

$$\begin{array}{ccc} \mathcal{IC}^\bullet_X[-2] & \xrightarrow{\alpha_{X,Y}} & \mathcal{IC}^\bullet_Y \\ {\rm Incl}_\# & \searrow & \downarrow \\ & & \mathcal{I}_\mathfrak{p}\mathcal{C}^\bullet_Y\,. \end{array}$$

The vertical arrow is the one induced by the inequality $\mathfrak{m} < \mathfrak{p}$. If $\alpha_{X,Y}$ is geometric, then it is compatible with homology. On $Y_{\rm reg}$ we have a geometric morphism $\alpha_{\rm reg} : \mathcal{IC}^\bullet_{X\cap Y_{\rm reg}}[-2] \to \mathcal{IC}^\bullet_{Y_{\rm reg}}$ which is the composition of the natural morphisms:

$$\mathcal{IC}^\bullet_{X\cap Y_{\rm reg}}[-2] \to \mathbb{D}^\bullet_{X\cap Y_{\rm reg}}[-2] \xrightarrow{{\rm incl}_\#} \mathbb{D}^\bullet_{Y_{\rm reg}} \cong \mathcal{IC}^\bullet_{Y_{\rm reg}}\,.$$

This is the unique morphism which is compatible with homology by construction. We are looking for an extension of α_{reg} to all Y. We will construct it for a case where the singularities have a special form.

Definition 1.1. Let $X \subset Y \subset M$ be a pair of analytic subvarieties of a smooth analytic manifold. We say that it has *conical singularities* if there exists a Whitney stratification of (Y, X) and for (each connected component of) each stratum S, there exists a normal slice N_S in M with coordinates such that $N_S \cap X$ and $N_S \cap Y$ are both given by homogeneous equations.

Examples of such spaces are Schubert varieties in Grassmannians. From the Hard Lefschetz of [BBD] we will deduce the following:

Theorem 1.2. *Let $X \subset Y$ be a pair of analytic varieties with conical singularities and* $\operatorname{codim}(X \subset Y) = 1$. *Suppose that no component of X is contained in the singularities of Y. Then there exists an unique extension $\alpha_{X,Y}$ of α_{reg} and this morphism is geometric.*

The case of general analytic singularities follows from M. Saito's theory of mixed Hodge modules. We give both proofs in §3. The unicity is in fact proved between the lines of [BBFGK], although it was not explicitly stated.

Remark 1.3. If a component of X is contained in the singularities of Y, then to construct a morphism of intersection homology, we would have to go through the normalization process of [BBFGK], p.166. At this point we would loose uniqueness.

2. Extending an Intersection Homology Morphism

The proof will be inductive. For the inductive step we fix some notation. We assume that the pair (Y, X) is stratified by a Whitney stratification. Let $\dim X = n$, $\dim Y = n + 1$. Let S be a minimal (connected) stratum of $X \setminus Y_{\text{reg}}$. Put $k = \operatorname{codim}(S \subset X)$. Let $X' = X \setminus S$ and $Y' = Y \setminus S$ and let $i : Y' \hookrightarrow Y$ be the inclusion. Then

$$\mathcal{IC}^\bullet_Y = \tau_{\leq k} R i_* \mathcal{IC}^\bullet_{Y'} ,$$

$$\mathcal{I}_{\mathfrak{p}}\mathcal{C}^\bullet_Y = \tau_{\leq k+1} R i_* \mathcal{I}_{\mathfrak{p}}\mathcal{C}^\bullet_{Y'} ,$$

and

$$\mathcal{IC}^\bullet_X = \tau_{\leq k-1} R i_* \mathcal{IC}^\bullet_{X'} .$$

Denote by $\mathcal{I}_{\underline{k}}\mathcal{C}^\bullet_Y$ another intersection homology sheaf on Y which is associated to the perversity

$$\underline{k}(2i) = \begin{cases} \mathfrak{m}(2i) = i - 1 & \text{for } i \leq k \\ \mathfrak{m}(2i) + 1 = i & \text{for } i > k. \end{cases}$$

Through the sheaf

$$\mathcal{I}_{\underline{k}}\mathcal{C}^\bullet_Y = \tau_{\leq k+1} R i_* \mathcal{IC}^\bullet_{Y'} ,$$

factors the canonical morphism $\mathcal{IC}^\bullet_Y \to \mathcal{I}_{\mathfrak{p}}\mathcal{C}^\bullet_Y$. Suppose we are given a morphism

$$\alpha_{X',Y'} : \mathcal{IC}^\bullet_{X'}[-2] \to \mathcal{IC}^\bullet_{Y'} .$$

Applying the functor $\tau_{\leq k+1}Ri_*$ to $\alpha_{X',Y'}$ we obtain a morphism $\phi : \mathcal{IC}^\bullet_X[-2] \to \mathcal{I}_{\underline{k}}\mathcal{C}^\bullet_Y$ as follows:

$$\begin{array}{cccc}
\tau_{\leq k+1}Ri_*\alpha_{X',Y'}: & \tau_{\leq k+1}Ri_*(\mathcal{IC}^\bullet_{X'}[-2]) & \to & \tau_{\leq k+1}Ri_*\mathcal{IC}^\bullet_{Y'} \\
 & \| & & \| \\
 & (\tau_{\leq k-1}Ri_*\mathcal{IC}^\bullet_{X'})[-2] & & \mathcal{I}_{\underline{k}}\mathcal{C}^\bullet_Y \\
 & \| & \nearrow \phi & \\
 & \mathcal{IC}^\bullet_X[-2] & &
\end{array}$$

Proposition 2.1. *The restriction of sheaves to Y' induces an isomorphism*

$$\mathrm{Hom}(\mathcal{IC}^\bullet_X[-2], \mathcal{I}_{\underline{k}}\mathcal{C}^\bullet_Y) \cong \mathrm{Hom}(\mathcal{IC}^\bullet_{X'}[-2], \mathcal{IC}^\bullet_{Y'})\,.$$

In particular, every extension of $\alpha_{X',Y'}$ equals ϕ.

This proposition follows from [BBFGK], Lemme 3.1. For the sake of completeness we will give a proof in our case.

Proof. We have

$$\begin{aligned}\mathrm{Hom}(\mathcal{IC}^\bullet_{X'}[-2], \mathcal{IC}^\bullet_{Y'}) &\cong \mathrm{Hom}(Ri^*\mathcal{IC}^\bullet_X[-2], \mathcal{IC}^\bullet_{Y'}) \cong \\ &\cong \mathrm{Hom}(\mathcal{IC}^\bullet_X[-2], Ri_*\mathcal{IC}^\bullet_{Y'})\,.\end{aligned}$$

Consider the distinguished triangle:

$$\begin{array}{ccc}
\mathcal{I}_{\underline{k}}\mathcal{C}^\bullet_Y & \longrightarrow & Ri_*\mathcal{IC}^\bullet_{Y'}\,. \\
{\scriptstyle [+1]} \nwarrow & & \swarrow \\
 & \tau_{\geq k+2}Ri_*\mathcal{IC}^\bullet_{Y'} &
\end{array}$$

Applying the functor $\mathrm{Hom}(\mathcal{IC}^\bullet_X[-2], \cdot\,)$, we obtain a long exact sequence:

$$\begin{aligned}&\to \mathrm{Hom}(\mathcal{IC}^\bullet_X[-2], \tau_{\geq k+2}Ri_*\mathcal{IC}^\bullet_{Y'}[-1]) \to \mathrm{Hom}(\mathcal{IC}^\bullet_X[-2], \mathcal{I}_{\underline{k}}\mathcal{C}^\bullet_Y) \to \\ &\qquad \to \mathrm{Hom}(\mathcal{IC}^\bullet_X[-2], Ri_*\mathcal{IC}^\bullet_{Y'}) \to \mathrm{Hom}(\mathcal{IC}^\bullet_X[-2], \tau_{\geq k+2}Ri_*\mathcal{IC}^\bullet_{Y'}) \to\end{aligned}$$

The cohomology of the sheaf $\mathcal{IC}^\bullet_X[-2]$ is concentrated in dimensions $2, 3, \ldots, k+1$. Thus the first and the fourth Hom–groups vanish, so in the middle we have an isomorphism. $\square$

Now we want to lift the morphism ϕ to $\mathcal{IC}^\bullet_Y$. To this end, we introduce more notation. Let N_S be a normal slice of S in M at some point $x \in S$. Then there exists a neighbourhood U of x such that $U \cap X \simeq B \times c\ell_X$ and $U \cap Y \simeq B \times c\ell_Y$, where B is a ball of the dimension $2(n-k)$ and $c\ell_X$ (resp. $c\ell_Y$) is the cone over the link ℓ_X (resp. ℓ_Y) of S in X (resp. Y). The morphism $\alpha_{X',Y'}$ induces a map for the k-th intersection homology of the links:

$$(\alpha_{\ell_X,\ell_Y})_k : IH_k(\ell_X) \to IH_k(\ell_Y)\,.$$

Proposition 2.2. *The morphism $\alpha_{X',Y'}$ extends to a morphism $\alpha_{X,Y}$ if and only if the map $(\alpha_{\ell_X,\ell_Y})_k$ vanishes. If $\alpha_{X,Y}$ exists, then it is unique. If $\alpha_{X',Y'}$ is geometric, then $\alpha_{X,Y}$ is geometric.*

Proof. We recall from [Bo], §1 that for a compact pseudomanifold L, we have

$$IH^{\mathrm{cld}}_i(cL) \cong \begin{cases} IH^{\mathrm{cld}}_i(cL \setminus \{*\}) \cong IH_{i-1}(L) & \text{for} \quad i > \frac{1}{2}\dim_{\mathbb{R}} cL \\ 0 & \text{for} \quad i \leq \frac{1}{2}\dim_{\mathbb{R}} cL \end{cases} \tag{2.3}$$

Assume that the extension $\alpha_{X,Y}$ exists. Then the map

$$(\alpha_{\ell_X,\ell_Y})_k : IH_k(\ell X) \to IH_k(\ell_Y)$$

may be completed to the commutative diagram which is induced by $\alpha_{X,Y}$ and the restriction from $c\ell_Y$ to $c\ell_Y \setminus \{x\}$:

$$\begin{array}{ccccccc} & & IH^{\rm cld}_{k+1}(c\ell_X) & \stackrel{\cong}{\to} & IH^{\rm cld}_{k+1}(c\ell_X \setminus \{x\}) & \cong & IH_k(\ell_X) \\ & & \downarrow & & \downarrow & & \downarrow (\alpha_{\ell_X,\ell_Y})_k \\ 0 & = & IH^{\rm cld}_{k+1}(c\ell_Y) & \to & IH^{\rm cld}_{k+1}(c\ell_Y \setminus \{x\}) & \cong & IH_k(\ell_Y)\,. \end{array} \tag{2.4}$$

The upper horizontal arrow is isomorphism and $IH^{\rm cld}_{k+1}(c\ell_Y) = 0$ by 2.3. Thus the map $(\alpha_{\ell_X,\ell_Y})_k$ vanishes.

To prove the converse, consider another distinguished triangle:

$$\begin{array}{ccc} \mathcal{IC}^\bullet_Y & \to & \mathcal{I}_{\underline{k}}\mathcal{C}^\bullet_Y\,, \\ {}_{[+1]}\nwarrow & & \swarrow \\ & \mathcal{H}^\bullet & \end{array}$$

where $\mathcal{H}^\bullet = \mathcal{H}^{k+1}Ri_*\mathcal{IC}^\bullet_{Y'}[-k-1]$ is the obstruction sheaf [GM1], §5.5. As in the proof of 2.1 above, the triangle induces a long exact sequence:

$$\to \mathrm{Hom}(\mathcal{IC}^\bullet_X[-2], \mathcal{H}^\bullet[-1]) \to \mathrm{Hom}(\mathcal{IC}^\bullet_X[-2], \mathcal{IC}^\bullet_Y) \to$$
$$\to \mathrm{Hom}(\mathcal{IC}^\bullet_X[-2], \mathcal{I}_{\underline{k}}\mathcal{C}^\bullet_Y) \to \mathrm{Hom}(\mathcal{IC}^\bullet_X[-2], \mathcal{H}^\bullet) \to$$
$$\phi \quad \mapsto \quad \mathrm{res}$$

The morphism ϕ factors through $\mathcal{IC}^\bullet_Y$ if and only if the resulting residue morphism

$$\mathrm{res} : \mathcal{IC}^\bullet_X[-2] \to \mathcal{H}^\bullet$$

vanishes. Note that in this case the lift $\mathcal{IC}^\bullet_X[-2] \to \mathcal{IC}^\bullet_Y$ of ϕ is unique (compare [BBFGK], p.167) : This is because the first term in the sequence vanishes as $\mathcal{IC}^\bullet_X[-2]$ is concentrated in dimensions $2, 3, \dots, k+1$ and the target sheaf only lives in dimension $k+2$. Moreover, by Proposition 2.1, any extension of $\alpha_{X',Y'}$ comes from ϕ.

Now we apply the functor $\tau_{\geq k+1}$ to the residue morphism and obtain the commutative square

$$\begin{array}{ccccc} & & \mathcal{IC}^\bullet_X[-2] & \xrightarrow{\mathrm{res}} & \mathcal{H}^\bullet \\ & & \downarrow & & \downarrow \cong \\ \mathcal{H}^{k+1}(\mathcal{IC}^\bullet_X[-2])[-k-1] & \cong & \tau_{\geq k+1}(\mathcal{IC}^\bullet_X[-2]) & \xrightarrow{\tau_{\geq k+1}\mathrm{res}} & \tau_{\geq k+1}\mathcal{H}^\bullet\,. \end{array}$$

We see that the residue morphism factors through $\mathcal{H}^{k+1}(\mathcal{IC}^\bullet_X[-2])[-k-1]$. The source and the target of $\tau_{\geq k+1}\mathrm{res}$ are sheaves which are concentrated in the dimension $k+1$. They are supported on S and are locally constant on S. It suffices to examine $\tau_{\geq k+1}\mathrm{res}$ at one points of S. The stalks on S of the sheaves considered here are:

$$\mathcal{H}^{k-1}(\mathcal{IC}^\bullet_X)_x = IH^{\rm cld}_{2\dim X-k+1}(U) \cong IH^{\rm cld}_{k+1}(c\ell_X) \cong IH_k(\ell_X)$$

and

$$\mathcal{H}^{k+1}(Ri_*\mathcal{IC}^\bullet_{Y'})_x = IH^{\rm cld}_{2\dim Y-k-1}(U\setminus S)\cong IH^{\rm cld}_{k+1}(c\ell_Y\setminus\{x\})\cong IH_k(\ell_Y)\,.$$

The morphism of the stalks is

$$(\alpha_{\ell_X,\ell_Y})_k : IH_k(\ell_X)\to IH_k(\ell_Y)\,.$$

When it vanishes, then the residue morphism vanishes as well.

If $\alpha_{X',Y'}$ is geometric then $\alpha_{X,Y}$ is geometric. This is because the sheaf $\mathcal{I}_{\underline{k}}\mathcal{C}^\bullet_Y$ maps to $\mathcal{I}_{\mathfrak{p}}\mathcal{C}^\bullet_Y$ and the composition with ϕ is the morphism $\mathrm{Incl}_\#$ as desired. □

Remark 2.5. To see some geometry hidden behind the sheaves consider the following situation: $n=k$, $S=\{x\}$, $Y\setminus\{x\}$ is smooth. We are given a cycle ξ which is allowable in X. Our goal is to make it allowable in Y. We have three cases:

1) If $\dim\xi\le n$, then ξ is not allowed to intersect $\{x\}$ in X, so it is allowable in Y.
2) If $\dim\xi\ge n+2$, then ξ is allowed to intersect $\{x\}$ both in X and Y.
3) A problem occurs if $\dim\xi=n+1$, since then ξ is allowed to intersect $\{x\}$ in X but not in Y. In a neighbourhood of x the cycle ξ is the cone over a cycle $\eta\subset\ell_X$. Suppose that $(\alpha_{\ell_X,\ell_Y})_k : IH_k(\ell_X)\to IH_k(\ell_Y)$ vanishes, i.e. η is a boundary in ℓ_Y, say $\eta=\partial\zeta$. Then the cycle $(\xi\setminus c\eta)\cup_\eta\zeta$ does not intersect $\{x\}$ and it is homologous with ξ in Y. This is the allowable cycle that we are looking for.

A similar consideration for isolated singularities one can find in [Y].

3. The Main Theorem

Let $\mathfrak{X}\subset\mathfrak{Y}\subset\mathbb{P}^N$ be a pair of projective varieties. Denote by $X\subset\mathbb{C}^{N+1}$ (resp. $Y\subset\mathbb{C}^{N+1}$) the affine cones over the $\mathfrak{X}$ (resp. $\mathfrak{Y}$) and by ℓ_X (resp. ℓ_Y) the links at $0\in\mathbb{C}^{N+1}$. We have a circle action on ℓ_X and ℓ_Y. Let $p:\ell_X\to\ell_X/S^1=\mathfrak{X}$ and $q:\ell_Y\to\ell_Y/S^1=\mathfrak{Y}$ be the quotients. We have isomorphisms:

$$Rq^*\mathcal{IC}^\bullet_{\mathfrak{X}}\cong\mathcal{IC}^\bullet_{\ell_X}\quad\text{and}\quad Rq^*\mathcal{IC}^\bullet_{\mathfrak{Y}}\cong\mathcal{IC}^\bullet_{\ell_Y}\,.$$

Proposition 3.1. *Let $\mathfrak{X}\subset\mathfrak{Y}\subset\mathbb{P}^N$ be a pair of projective varieties with $\dim\mathfrak{X}=k-1$ and $\dim\mathfrak{Y}=k$. Suppose that there is given a morphism of intersection homology $\alpha_{\mathfrak{X},\mathfrak{Y}}:\mathcal{IC}^\bullet_{\mathfrak{X}}[-2]\to\mathcal{IC}^\bullet_{\mathfrak{Y}}$. Then the induced map of intersection homology $(\alpha_{\ell_X,\ell_Y})_k : IH_k(\ell_X)\to IH_k(\ell_Y)$ vanishes.*

Proof. We have a morphism of Gysin sequences of the fibrations p and q induced by $\alpha_{\mathfrak{X},\mathfrak{Y}}$:

$$\begin{array}{ccccccccc}
IH_{k+1}(\mathfrak{X}) & \overset{\Lambda}{\hookrightarrow} & IH_{k-1}(\mathfrak{X}) & \overset{p^*}{\twoheadrightarrow} & IH_k(\ell_X) & \underset{0}{\overset{p_*}{\to}} & IH_k(\mathfrak{X}) & \underset{\cong}{\overset{\Lambda}{\to}} & IH_{k-2}(\mathfrak{X}) \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle(\alpha_{\ell_X,\ell_Y})_k} & & \downarrow & & \downarrow \\
IH_{k+1}(\mathfrak{Y}) & \underset{\cong}{\overset{\Lambda}{\to}} & IH_{k-1}(\mathfrak{Y}) & \underset{0}{\overset{q^*}{\to}} & IH_k(\ell_Y) & \overset{q_*}{\hookrightarrow} & IH_k(\mathfrak{Y}) & \overset{\Lambda}{\twoheadrightarrow} & IH_{k-2}(\mathfrak{Y})\,.
\end{array}$$

The map Λ is given by intersecting with the Chern class of the tautological bundle, i.e., with the hyperplane section. The Hard Lefschetz theorem, which is valid for intersection homology by [BBD], asserts that Λ is an isomorphism in both cases indicated by $\cong$. Thus p_* and q^* vanish, so p^* is surjective and q_* is injective. Analyzing the diagram, we conclude that the map $(\alpha_{\ell_X,\ell_Y})_k$ vanishes. $\square$

Remark 3.2. Modifying our method of proof we can generalize Proposition 3.1 for quasihomogeneous singularities.

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. The proof is an induction on $\dim X$. For $\dim X = 0$, the theorem is obvious.

Suppose that $\dim X = n$. We want to construct a morphism $\alpha_{X,Y}$ which is an extension of $\alpha_{\rm reg}$. We do it stratum by stratum going down with respect to the partial ordering among the strata. To extend it over S, we choose a slice N_S given by the definition of conical singularities. By the inductive assumption there exists a morphism of intersection homology $\alpha'_{\mathfrak{X},\mathfrak{Y}} : \mathcal{IC}^\bullet_{\mathfrak{X}}[-2] \to \mathcal{IC}^\bullet_{\mathfrak{Y}}$, where $\mathfrak{X} = \ell_X/S^1$ and $\mathfrak{Y} = \ell_Y/S^1$. By the uniqueness, the morphism $Rq^*\alpha'_{\mathfrak{X},\mathfrak{Y}}$ coincides with $\alpha_{\ell_X,\ell_Y} = \alpha_{X',Y'|\ell_Y}$. (Here $q : \ell_Y \to \mathfrak{Y}$ is again the quotient map.) By Proposition 3.1, the map $(\alpha_{\ell_X\ell_Y})_k$ vanishes. Thus by Proposition 2.2, an extension exists and is unique. Since $\alpha_{\rm reg}$ was geometric, the obtained extension is geometric as well. $\square$

In the end of this section we show how to deduce the existence of a morphism $\alpha_{X,Y}$ for a pair of analytic singularities. Our argument is based on the theory of mixed Hodge modules due to M. Saito. The proof is modified version of [We].

Theorem 3.3. *Assume that (Y, X) is a pair of analytic varieties with codimension* $\operatorname{codim}(X\subset Y) = 1$ *and suppose that no component of X is contained in the singularities of Y. Then there exists an unique extension $\alpha_{X,Y}$ of $\alpha_{\rm reg}$.*

Proof. The question about the existence of $\alpha_{X,Y}$ has a local nature. If it exists locally, then the morphism of links vanishes. Then, by Proposition 2.2, a global morphism exists, it is unique and locally it coincides with the original local ones. Thus we can always patch morphisms. We will construct $\alpha_{X,Y}$ locally. Analyzing the techniques of resolution of singularities, we note that for each $x \in X$ one can find a neighbourhood U in Y and a resolution $\pi : \tilde{U} \to U$, which is a projective morphism, i.e. it factors through an inclusion and the projection

$$\tilde{U} \hookrightarrow U \times \mathbb{P}^N \twoheadrightarrow U\,.$$

The decomposition theorem is valid for π by [S2], 0.3 or 0.6. Hence $\mathcal{IC}^\bullet_U$ is a direct summand in $R\pi_*\mathbb{D}^\bullet_{\tilde{U}}$. Analogously $\mathcal{IC}^\bullet_{U\cap X}$ is a direct summand in $R\pi_*\mathcal{IC}^\bullet_{\pi^{-1}(U\cap X)}$. Then the composition

$$\mathcal{IC}^\bullet_{U\cap X}[-2] \hookrightarrow R\pi_*\mathcal{IC}^\bullet_{\pi^{-1}(U\cap X)}[-2] \xrightarrow{R\pi_*\mathrm{incl}_\#} R\pi_*\mathbb{D}^\bullet_{\tilde{U}} \twoheadrightarrow \mathcal{IC}^\bullet_U$$

is an extension of $\alpha_{\rm reg}$ after suitable rescaling. $\square$

Remark 3.4. As before if we allow X to lie in the singularities of Y, then $\alpha_{X,Y}$ exists, but it depends on the choice made on $X_{\rm reg}$. In fact we can proceede as follows. Suppose X equals the closure of a singular stratum $Y_\alpha \subset Y$. Let $L_\alpha = \mathcal{H}^0(\mathcal{IC}^\bullet_Y)_{|Y_\alpha}$ be the cohomology sheaf. It is locally constant, i.e. it is a coefficient system on Y_α. Then we obtain a morphism $\mathcal{IC}^\bullet_X(L^*_\alpha)[-2] \to \mathcal{IC}^\bullet_Y$, which is an unique extension of the dual of the restriction morphism on $Y_{\rm reg} \cup Y_\alpha$:

$$\mathcal{IC}^\bullet_{Y_{\rm reg}\cup Y_\alpha} \to \mathcal{IC}^\bullet_{Y_\alpha}(L_\alpha) \cong L_\alpha\,.$$

4. Example – Projective Cones

Let $\mathfrak{X} \subset \mathfrak{Y} \subset \mathbb{P}^N$ be a pair of complex projective varieties, $\dim \mathfrak{X} = n-1$, $\dim \mathfrak{Y} = n$. Let $K\mathfrak{X} \subset \mathbb{P}^{N+1}$ and $K\mathfrak{Y} \subset \mathbb{P}^{N+1}$ be the projective cones over $\mathfrak{X}$ and $\mathfrak{Y}$. Topologically they are the Thom spaces of the tautological bundle restricted to $\mathfrak{X}$ and $\mathfrak{Y}$. They are also called algebraic suspensions. Suppose we are given a map

$$(\alpha_{\mathfrak{X},\mathfrak{Y}})_* : IH_*(\mathfrak{X}) \to IH_*(\mathfrak{Y})\,.$$

A question arises : What is the map

$$(\alpha_{K\mathfrak{X},K\mathfrak{Y}})_* : IH_*(K\mathfrak{X}) \to IH_*(K\mathfrak{Y})\,,$$

which comes from the continuation of $\alpha_{\mathfrak{X},\mathfrak{Y}}$? We give an answer in the following table. The intersection homology of a projective cone is computed in [FK], 3.5. Let $x_i = IH_i(\mathfrak{X})$ and $y_i = IH_i(\mathfrak{Y})$.

dim		$IH_*(K\mathfrak{X})$	map	$IH_*(K\mathfrak{Y})$	
0	$*$	x_0	$(\alpha_{\mathfrak{X},\mathfrak{Y}})_0$	y_0	$*$
1	$*$	x_1	$(\alpha_{\mathfrak{X},\mathfrak{Y}})_1$	y_1	$*$
...	$*$				$*$
$n-1$	$*$	x_{n-1}	$(\alpha_{\mathfrak{X},\mathfrak{Y}})_{n-1}$	y_{n-1}	$*$
n	$*$	$\mathrm{im}(\Lambda : x_n \xrightarrow{\cong} x_{n-2})$	$(\overline{\alpha_{\mathfrak{X},\mathfrak{Y}}})_n$	y_n	$*$
$n+1$	K	x_{n-1}	$\Lambda^{-1}(\alpha_{\mathfrak{X},\mathfrak{Y}})_{n-1}$	$\mathrm{im}(\Lambda : y_{n+1} \xrightarrow{\cong} y_{n-1})$	$*$
$n+2$	K	x_n	$(\alpha_{\mathfrak{X},\mathfrak{Y}})_n$	y_n	K
...	K				K
$2n$	K	x_{2n-2}	$(\alpha_{\mathfrak{X},\mathfrak{Y}})_{2n-2}$	y_{2n-2}	K
$2n+1$		0		y_{2n-1}	K
$2n+2$		0		y_{2n}	K

The star $*$ in the table indicates, that the classes of the corresponding intersection homology group are represented by the cycles contained in the base of the cone otherwise they are represented by the cones over the cycles. In the dimension n the map $(\alpha_{\mathfrak{X},\mathfrak{Y}})_n$ defines a map $(\overline{\alpha_{\mathfrak{X},\mathfrak{Y}}})_n : \mathrm{im}(\Lambda : x_n \xrightarrow{\cong} x_{n-2}) \to y_n$. In the dimension $n+1$ we apply $(\alpha_{\mathfrak{X},\mathfrak{Y}})_{n-1}$ to a cycle from x_{n-1} and then apply the inverse of Λ to obtain an allowable representative in y_{n+1}.

5. Vanishing in Homology of Link Mappings

Goresky and MacPherson [GM2] list homological properties of algebraic varieties which follow from the decomposition theorem of [BBD]. We will add to this list another one. Proposition 2.2 says that a remarkable property of links is responsible for the continuation of $\alpha_{X',Y'}$ on S. By Theorem 3.3 and Remark 3.4 such a continuation exists for arbitrary analytic varieties.

Corollary 5.1. *Let (Y, X) be a pair of analytic varieties with* $\mathrm{codim}(X \subset Y) = 1$. *Let ℓ_X and ℓ_Y be the links of a stratum of codimension k in X and let $\alpha_{X\setminus\overline{S},Y\setminus\overline{S}} : \mathcal{IC}^\bullet_{X\setminus\overline{S}}[-2] \to \mathcal{IC}^\bullet_{Y\setminus\overline{S}}$ be any morphism of intersection homology. Then the induced map $(\alpha_{\ell_X,\ell_Y})_k : IH_k(\ell_X) \to IH_k(\ell_Y)$ vanishes.*

In the algebraic case the argument of [BBFGK], p. 167 and Lemme clef 3.3, is to show vanishing of certain sheaf morphism which directly implies Corollary 5.1. For a pair of analytic varieties of codimension $d > 1$ a morphism $\alpha_{X,Y} : \mathcal{IC}^\bullet_X[-2d] \to \mathcal{IC}^\bullet_Y$ can be constructed locally as in the proof of 3.3. (There is no guarantee that they come from a global morphism.) The diagram 2.4 proves vanishing as before. For isolated singularities, we obtain:

Corollary 5.2. *Let $X \subset Y$ be a pair of analytic varieties with $\dim X = n$ and $\dim Y = n + d$. Suppose that x is an isolated singular point of X and Y. Let $\ell_X \subset \ell_Y$ be its links. Then the maps induced by the inclusion $\iota_* : H_i(\ell_X) \to H_i(\ell_Y)$ vanish for $n \le i < n + d$.*

Corollary 5.2 is related to the results of [HL], although it is of different nature. It can be derived from the fact, that $H^*(\ell_X)$ and $H^*(\ell_Y)$ are equipped with the mixed Hodge structure, see e.g. [Di], C28:

$$H^i(\ell_X) \text{ has all the weights } \begin{cases} \le i \text{ for } i < n \\ > i \text{ for } i \ge n\,, \end{cases}$$

$$H^i(\ell_Y) \text{ has all the weighs } \begin{cases} \le i \text{ for } i < n + d \\ > i \text{ for } i \ge n + d\,. \end{cases}$$

The map $\iota^* : H^*(\ell_Y) \to H^*(\ell_X)$ preserves mixed Hodge structure, thus it has to vanish in the range $n \le i < n + d$, as noticed in the proof of [Di], C34. For nonisolated singularities the analogous statement is also true. By [S1], 1.18 or [DS] the intersection cohomology of links is equipped with mixed Hodge structure with the weights as above. If $(\alpha_{\ell_X,\ell_Y})^*$ preserves the weights, then it has to vanish in the middle.

Remark 5.3. If L is a local system defined on an open set $U \subset Y_{\text{reg}}$ and no component of X is contained in $Y \setminus U$, then by standard operation on sheaves (as in Section 2) we obtain a morphism of intersection homology with twisted coefficients:

$$\text{Incl}_\# : \mathcal{IC}^\bullet_X(L_{|X\cap U})[-2] \to \mathcal{I}_\mathfrak{p}\mathcal{C}^\bullet_Y(L)\,.$$

Suppose that a local system has geometric origin ([BBD], p. 162 or [S2], p. 128) then modifying the proof of [BBFGK] or [We] one can show generalized Corollaries 5.1–2 for twisted coefficients. This way we can prove the existence of a morphism

$$(\alpha_{X,Y})_* : \mathcal{IC}^\bullet_X(L_{|X\cap U})[-2] \to \mathcal{IC}^\bullet_Y(L)\,.$$

The Corollaries 5.1–2 we consider as a "local Hard Lefschetz" theorem. The reason for this is following:

Proposition 5.4. *The Corollary 5.1 implies the Hard Lefschetz theorem for intersection homology.*

Proof. Let $\mathfrak{Y}$ be a projective variety of dimension n and let $\mathfrak{X} = \mathfrak{Y} \cap H$ be a generic hyperplane section. The inclusion $i : \mathfrak{X} \hookrightarrow \mathfrak{Y}$ is normally nonsingular, thus it induces a map of intersection homology. Consider again the diagram with Gysin sequences of 3.1:

$$\begin{array}{ccccccccc} \longrightarrow & IH_{n+1}(\mathfrak{X}) & \xrightarrow{\Lambda} & IH_{n-1}(\mathfrak{X}) & \xrightarrow{p^*} & IH_n(\ell_X) & \longrightarrow \\ & \downarrow{\scriptstyle i_{n+1}} & & \downarrow{\scriptstyle i_{n-1}} & & \downarrow{\scriptstyle 0} & \\ \longrightarrow & IH_{n+1}(\mathfrak{Y}) & \xrightarrow{\Lambda} & IH_{n-1}(\mathfrak{Y}) & \xrightarrow{q^*} & IH_n(\ell_Y) & \longrightarrow & . \end{array}$$

By the weak Lefschetz theorem [GM3], §6.10 the map i_{n-1} is surjective. Thus if the map of links vanishes in the middle homology, then q^* vanishes as well. We conclude that Λ for $\mathfrak{Y}$ is surjective. By Poincaré duality it is an isomorphism. The argument that $\Lambda^k : IH_{n+k}(\mathfrak{Y}) \to IH_{n-k}(\mathfrak{X})$ is an isomorphism for $k > 1$ is standard; it follows from the weak Lefschetz theorem. □

6. Application – Chern Classes

Let X be a n–dimensional analytic variety. Suppose we are given a sequence of subvarieties:

$$\mathcal{F} = \{X = X^0 \supset X^1 \supset \cdots \supset X^k\}.$$

Assume that no component of X^i is contained in the singularities of X^{i-1} and $\operatorname{codim}(X^i \subset X) = i$ for $i = 1, 2, \ldots, k$. We will call such sequence *a flag* in X. With a flag in X, we associate a sequence of elements in intersection homology: $[\mathcal{F}]^i \in IH^{\mathrm{cld}}_{2(n-i)}(X)$. These classes are lifts of the corresponding fundamental classes $[X^i] \in H^{\mathrm{cld}}_{2(n-i)}(X)$. The construction of $[\mathcal{F}]^i$ is the following : We take the fundamental class of X^i in $IH^{\mathrm{cld}}_{2(n-i)}(X^i)$ and we lift it step by step going through $IH^{\mathrm{cld}}_{2(n-i)}(X^j)$ for $0 \le j < i$. On each step we choose the lift induced by $(\alpha_{X^j, X^{j-1}})_*$.

Example 6.1. Let $\mathfrak{X} \subset \mathbb{P}^N$ be a projective variety of pure dimension n and let $c\ell_X \subset \mathbb{C}^{N+1}$ be its affine cone. Consider a flag of linear subspaces

$$\mathcal{V} = \{V^{n+1} \subset V^n \subset \cdots \subset V^1 \subset V^0 = \mathbb{C}^{N+1}\}$$

with $\operatorname{codim}(V^i \subset \mathbb{C}^{N+1}) = i$ for $i = 0, 1, \ldots, n+1$. Let

$$p_i : c\ell_X \to \mathbb{C}^{N+1}/V^{n+2-i} \cong \mathbb{C}^{n+2-i}$$

be the linear projection restricted to $c\ell_X$. Consider the set of critical points of p_i

$$(c\ell_X)^i = \operatorname{closure}\{x \in (c\ell_X)_{\mathrm{reg}} : p_i \text{ is not submersion at } x\},$$

where $(c\ell_X)_{\mathrm{reg}}$ is the nonsingular part of $c\ell_X$. Each set $(c\ell_X)^i$ is homogeneous. We denote its projectivization by $\mathfrak{X}^i$. We call it the i-th polar variety of $\mathfrak{X}$. We have $\mathfrak{X}^i \subset \mathfrak{X}^{i-1}$ for $0 < i \le n+1$ and we put $\mathfrak{X}^0 = \mathfrak{X}$. If the flag $\mathcal{V}$ is general enough, then the sequence

$$\mathcal{P}_{\mathcal{V}} = \{\mathfrak{X}^0 = \mathfrak{X} \supset \mathfrak{X}^1 \supset \cdots \supset \mathfrak{X}^n\}$$

is a flag in $\mathfrak{X}$ and $\mathfrak{X}^{n+1} = \emptyset$. We obtain classes $[\mathcal{P}_{\mathcal{V}}]^i \in IH_{2(n-i)}(\mathfrak{X})$. If $\mathfrak{X}$ is smooth, then $(-1)^i[\mathcal{P}_{\mathcal{V}}]^i = (-1)^i[\mathfrak{X}^i] \in IH_{2(n-i)}(\mathfrak{X}) \cong H_{2(n-i)}(\mathfrak{X})$ is the Poincaré dual of the Chern class of a bundle Θ, where Θ fits to an exact sequence of bundles:

$$0 \to \mathcal{O}(-1) \to \Theta \to T\mathfrak{X} \otimes \mathcal{O}(-1) \to 0,$$

see [Po]. Then the duals of the Chern classes of $\mathfrak{X}$ satisfy the formula:

$$c_i(\mathfrak{X}) = \sum_{j=0}^{i} \binom{n+1-j}{i-j} (-1)^j h^{i-j} \cap [\mathfrak{X}^j], \tag{6.2}$$

where $h \in H^2(\mathfrak{X})$ is the class of the hyperplane section, compare [Fu], §14.4.15. When $\mathfrak{X}$ is singular, then by [Pi] the formula 6.1 describes the Chern-Mather class : $c_i^M(\mathfrak{X}) \in H_{2(n-i)}(\mathfrak{X})$ which in general does not lie in the image of the Poincaré duality map. By the very same formula with $[\mathfrak{X}^j]$ replaced by $[\mathcal{P}_{\mathcal{V}}]^j$ we define an element in intersection homology. This is a particular lift of $c_i^M(\mathfrak{X})$ to intersection homology. We conjecture that it does not depend on the embedding.

Remark 6.3. This method of defining characteristic classes was already known in the thirties. This is the way how Todd in [Td] defined a "canonical system" with the hyperplane bundle replaced by an arbitrary bundle. It was later called the Chern class of $\mathfrak{X}$. There are enough evidences that polar varieties carry an important information about global and local invariants of $\mathfrak{X}$, see e.g. [LT]. The Chern classes in intersection homology were defined by J-P. Brasselet and by the author. The paper [BW] contains a proof that they do not depend on the choice of the generic flag $\mathcal{V}$. It also contains an explicit computation for $\mathfrak{X}$ which is the projective cone over a quadric in $\mathbb{P}^3$.

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