CORRESPONDENCES AND INDEX

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Abstract. We define certain class of correspondences of polarized representations of C^*-algebras. Our correspondences are modeled on the spaces of boundary values of elliptic operators on bordisms joining two manifolds. In this setup we define the index. The main subject of the paper is the additivity of the index.

Mathematics Subject Classification (2000). Primary: 58J32, Secondary: 35J55, 35Q15, 47A53, 55N15, 58J20.

Keywords. Index of an elliptic operator, Riemann–Hilbert problem, bordism, K-theory.

1. Introduction

Let X be a closed manifold. Suppose it is decomposed into a sum of two manifolds X_+, X_- glued along the common boundary

$$\partial X_+ = \partial X_- = M\,.$$

Let

$$D : C^\infty(X;\xi) \to C^\infty(X;\eta)$$

be an elliptic operator of the first order. We assume that it possesses the unique extension property: if $Df = 0$ and $f_{|M} = 0$ then $f = 0$. In what follows we will consider only elliptic operators of the first order such that D and D^* have the unique extension property.

One defines the spaces $H_\epsilon(D) \subset L^2(M;\xi)$ for $\epsilon \in \{+,-\}$, which are the closures of the spaces of boundary values of solutions of $Df = 0$ on the manifolds X_ϵ with boundary $\partial X_\epsilon = M$. The space $H_\epsilon(D)$ is defined to be the closure of :

$$\{f \in C^\infty(M;\xi)\,:\,\exists \tilde f \in C^\infty(X_\epsilon;\xi),\ f = \tilde f_{|M},\ \ D(\tilde f) = 0\,\}$$

in $L^2(M;\xi)$. The pair of spaces $H_\pm(D)$ is a Fredholm pair, [4]. There are associated Calderón projectors $P_+(D)$ and $P_-(D)$, see [19].

A.W. is supported by KBN grant 1 P03A 005 26.

To organize somehow the set of possible Cauchy data we will introduce certain algebraic object. We fix a C^*-algebra B, which is the algebra of functions on M in our case. Suppose it acts on a Hilbert space H. Now we consider Fredholm pairs in H. In our case $H = L^2(M;\xi)$ and one of the possible Fredholm pairs is $H_\pm(D)$. Note that this pair is not arbitrary. It has a property which we called *good*. A Fredholm pair is good if (roughly speaking) it remains to be Fredholm after conjugation with functions, see §4. These pairs act naturally on $K_1(M)$. Nevertheless the concept of a good Fredholm pair is not convenient to manipulate, thus we restrict our attention to the pairs of geometric origin, see §5. We call them *admissible*. They are the pairs of subspaces which are images of projectors which almost commute with the actions of the algebra B. This concept allows to extract the relevant analytico–functional information out of the Cauchy data. Further a Morse decomposition of a manifold is translated into this language.

Our paper is devoted to the study of the cut and paste technique on manifolds and its effect on indices. The spirit of these constructions comes from the earlier papers [4]–[6] or [8]. According to the topological and conformal field theory we investigate the behaviour of the index of a differential operator on a manifold composed from bordisms

$$X = X_0 \cup_{M_1} X_1 \cup_{M_2} \cdots \cup_{M_{m-1}} X_{m-1} \cup_{M_m} X_m \,.$$

We think of M_i's as objects and we treat bordisms of manifolds as morphisms. Starting from this geometric background we introduce a category **PR**, whose objects are *polarized representations*. The algebra B may vary. We keep in mind that such objects arise when:

- B is an algebra of functions on a manifold M,
- there is given a vector bundle ξ over M, then $H = L^2(M;\xi)$ is a representation of B,
- there is given a pseudodifferential projector in H.

The morphisms in **PR** are certain correspondences, i.e. linear subspaces in the product of the source and the target. A particular case of principal value for our theory are the correspondences coming from bordisms of manifolds equipped with an elliptic operator. Precisely: suppose we are given a manifold W with a boundary $\partial W = M_1 \sqcup M_2$. Moreover, suppose that there is given an elliptic operator of the first order acting on the sections of a vector bundle ξ over W. Then the space of the boundary values of the Cauchy data of solutions is a linear subspace in $L^2(M_1;\xi_{|M_1}) \oplus L^2(M_2;\xi_{|M_2})$. In another words it is a correspondence from $L^2(M_1;\xi_{|M_1})$ to $L^2(M_2;\xi_{|M_2})$.

Basic example: The following example is instructive and serves as the model situation (see [7]): Let $W = \{z \in \mathbf{C} \,:\, r_1 \geq |z| \geq r_2\}$ be a ring domain and let D be the Cauchy-Riemann operator. The space $L^2(M_i)$ for $i = 1, 2$ is identified with the space of sequences $\{a_n\}_{n\in\mathbf{Z}}$, such that $\sum_{n\in\mathbf{Z}} |a_n|^2 r_i^{2n} < \infty$. The sequence $\{a_n\}$ defines the function on M_i given by the formula $f(z) = \sum_{n\in\mathbf{Z}} a_n z^n$. The subspace of the boundary values of holomorphic functions on W is identified with

$$\Big\{(\{a_n\},\{b_n\})\ :\ \Sigma_{n\in\mathbf{Z}}|a_n|^2 r_1^{2n} < \infty\,,\ \Sigma_{n\in\mathbf{Z}}|b_n|^2 r_2^{2n} < \infty \text{ and } a_n = b_n\Big\}\,.$$

It can be treated as the graph of an unbounded operator $\Phi : L^2(M_1) \to L^2(M_2)$. When we restrict Φ to the space $L^2(M_1)^\sharp$ consisting of the functions with coefficients $a_n = 0$ for $n < 0$ we obtain a compact operator. On the other hand the inverse operator $\Phi^{-1} : L^2(M_2) \to L^2(M_1)$ is compact when restricted to $L^2(M_2)^\flat$, the space consisting of the functions with coefficients $a_n = 0$ for $n \geq 0$.

The Riemann-Hilbert transmission problem of the Cauchy data across a hypersurface is a model for another class of morphisms. These are called *twists*. Our approach allows us to treat bordisms and twists in a uniform way. We calculate the global index of an elliptic operator in terms of local indices depending only on the pieces of the decomposed manifold (see Theorems 9.6 and 11.1). An interesting phenomenon occurs. The index is not additive with respect to the composition of bordisms. Instead each composition creates a contribution to the global index (Theorem 10.2):

$$L_1,\, L_2 \rightsquigarrow L_2 \circ L_1 + \delta(L_1, L_2)\,.$$

In the geometric situation this contribution might be nonzero for example when a closed manifold is created as an effect of composition of bordisms. One can show that if the bordisms in **PR** come from connected geometric bordisms supporting elliptic operators with the unique extension property then the index is additive. The contributions coming from twists are equivalent to the effects of pairings in the odd K-theory, Theorem 9.7.

It's a good moment now to expose a fundamental role of the splitting of the Hilbert space into a direct sum. The need of introducing a splitting was clear already in [4]:

- It was used to the study of Fredholm pairs with application to the Riemann-Hilbert transmission problem in [4]
- Splitting also came into light in the paper of Kasparov [13], who introduced a homological K-theory built from the Hilbert modules. The program of noncommutative geometry of A.Connes develops this idea, [10, 11] .
- Splitting plays an important role in the theory of loop groups in [16].
- There is also a number of papers in which surgery of the Dirac operator is studied. Splitting serves as a boundary condition, see e.g. [12], [17]. These papers originate from [2].

In the present paper we omit the technicalities and problems arising for a general elliptic operator. We concentrate on the purely functional calculus of correspondences. This is mainly the linear algebra.

2. Fredholm pairs

Let us first summarize some facts about Fredholm pairs. We will follow [4]-[6]. Suppose that H_+ and H_- are two closed subspaces of a Hilbert space, such that $H_+ + H_-$ is also closed and

- $H_+ \cap H_-$ is of finite dimension,
- $H_+ + H_-$ is of finite codimension.

We assume that both spaces have infinite dimension. Then we say that the pair $(H_+, H_-) = H_\pm$ is Fredholm. We define its index

$$Ind(H_\pm) = \dim(H_+ \cap H_-) - \operatorname{codim}(H_+ + H_-)\,.$$

The following statements follow from easy linear algebra.

Proposition 2.1. *A pair $H_\pm$ is Fredholm if, and only if the map*

$$\iota : H_+ \oplus H_- \to H$$

induced by the inclusions is a Fredholm operator. Moreover the indices are equal:

$$Ind(H_\pm) = ind(\iota)\,.$$

Here Ind denotes the index of a pair, whereas ind stands for the index of an operator. Suppose that H is decomposed into a direct sum

$$H = H^\flat \oplus H^\sharp\,.$$

We may assume that this decomposition is given by a symmetry S: a "sign" or "signature" operator. Let $P^\flat$ and $P^\sharp$ be the corresponding projectors. We can write $S = P^\sharp - P^\flat$. We easily have:

Proposition 2.2. *If $H_\pm$ is a pair with $H_+ = H^\sharp$, then it is Fredholm if and only if the restriction $P^\flat_{|H_-} : H_- \to H^\flat$ is a Fredholm operator. Moreover the indices are equal:*

$$Ind(H_\pm) = ind(P^\flat_{|H_-})\,.$$

Let $I \subset L(H)$ be an ideal which is lies between the ideal of finite rank operators and the ideal of compact operators

$$F \subset I \subset K\,.$$

Define $GL(P^\flat, I) \subset GL(H)$ to be the set of the invertible automorphisms of H commuting with $P^\flat$ up to the ideal I. We will say that ϕ almost commutes with $P^\flat$ or we will write $\phi P^\flat \sim P^\flat\phi$. Obviously $GL(P^\flat, I) = GL(P^\sharp, I) = GL(S, I)$. We have the following description of Fredholm pairs stated in [4]. (The proof is again an easy linear algebra.)

Theorem 2.3. *Let $H_\pm$ be a Fredholm pair with $H_+ = H^\sharp$. Then there exists a complement $H^\flat$ (that is $H^\flat \oplus H^\sharp = H$) and there exists $\phi \in GL(P^\flat, I)$, such that $H_- = \phi(H^\flat)$. If $H_\pm$ is given by a pair of projectors $P_\pm$ satisfying $P_- + P_- - 1 \in I$, then we can take $H^\flat = ker\, P_+$. Moreover, the operator $\phi P^\flat + P^\sharp$ is Fredholm and*

$$ind(\phi P^\flat + P^\sharp) = Ind(H_\pm)\,.$$

The map

$$\widetilde{ind} : GL(P^\flat, I) \to \mathbf{Z}$$

$$\widetilde{ind}(\phi) = ind(\phi P^\flat + P^\sharp)$$

is a group homomorphism.

It follows that

$$ind(\phi P^\flat + P^\sharp) = ind(P^\flat \phi : H^\flat \to H^\flat) = ind(P^\sharp \phi^{-1} : H^\sharp \to H^\sharp)\,.$$

3. Index formula for a decomposed manifold

The main example of a Fredholm pair is the following. Let D be an elliptic operator on $X = X_+ \cup_M X_-$. Then the pair of boundary value spaces $H_\pm(D)$ (as defined in the introduction) is a Fredholm pair.

Assumption 3.1 (Unique Extension Property). Let $\epsilon = +$ or $-$ and let $f \in C^\infty(X_\epsilon; \xi)$. If $Df = 0$ and $f_{|M} = 0$ then $f = 0$.

If D has the unique extension property, then

$$\ker(D) \simeq H_+(D) \cap H_-(D)\,.$$

This formula is easy to explain: a global solution restricted to M lies in $H_+(D) \cap H_-(D)$. On the other hand if a section f of ξ over M can be extended to both X_+ and X_-, such that the extensions are solutions of $Df = 0$ then we can glue them to obtain a global solution. The unique extension property is necessary, because we need to know that a solution is determined by its restriction to M. Following the reasoning in [4], with the Assumption 3.1 for D and D^* we have:

Corollary 3.2.

$$Ind(H_\pm(D)) = ind(D)\,.$$

For a rigorous proof see [9], §24 for Dirac type operators.

Remark 3.3. It may happen that D does not have the unique extension property. This is so for example when X is not connected. Then the Cauchy data $H_\pm(D)$ do not say anything about the index of the operator D on the components of X disjoined with M. There are also known elliptic operators without the unique extension property on connected manifolds, [15], [1]. It is difficult to characterize the class of all operators D with the unique extension property. Nevertheless the most relevant are Cauchy-Riemann and Dirac type operators. These operators have the unique extension property.

4. Good Fredholm pairs

Suppose there is given an algebra B and its representation ρ in a Hilbert space H. For a Fredholm pair $H_\pm$ in H and an invertible matrix $A \in GL_n(B)$ we define a new pair of subspaces $A \bowtie H_\pm$ in $H^{\oplus n}$. We set

$$(A \bowtie H_\pm)_- = \rho A(H_-^{\oplus n}) \qquad (A \bowtie H_\pm)_+ = H_+^{\oplus n}\,.$$

(As usually we treat ρA as an automorphism of $H^{\oplus n}$.)

Definition 4.1. Let B be a C^*-algebra which acts on a Hilbert space H. A *good Fredholm pair* is a pair of subspaces (H_+, H_-) in H, such that for any invertible matrix $A \in GL(n; B)$ the pair $A \bowtie H_\pm$ is a Fredholm pair.

We will see that the pair of boundary values $H_\pm(D) \subset H = L^2(M; \xi)$ for the operator D considered in the introduction is good.

Example. [Main example: Riemann-Hilbert problem] Consider the following problem: there is given a matrix-valued function $A : M \to GL_n(\mathbf{C})$. We look for the sequence $(s^1_\pm, \ldots, s^n_\pm)$ of solutions of $Ds = 0$ on $X_\pm$ satisfying the transmission condition on M

$$A(s^1_-, \ldots, s^n_-) = (s^1_+, \ldots, s^n_+)\,.$$

A Fredholm operator is related to this problem and we study its index, see §11. On the other hand the matrix A treated as the gluing data defines an n-dimensional vector bundle Θ^A_X over X. Then

$$Ind(A \bowtie H_\pm(D)) = ind(D \otimes \Theta^A_X)\,.$$

This formula was obtained in [8], §1 under the assumption that D has a product form along M.

Corollary 4.2. *For the elliptic operator D the pair $H_\pm(D) \subset L^2(M; \xi)$ is a good Fredholm pair.*

Remark 4.3. Consider the differential in the Mayer-Vietoris exact sequence of $X = X_+ \cup_M X_-$

$$\delta : K_0(X) \to K_{-1}(M)\,.$$

The operator D defines a class $[D] \in K_0(X)$. The element $\delta[D]$ can be recovered from the good Fredholm pair $H_\pm(D) \subset L^2(M; \xi)$. Note that the pair $H_\pm(D)$ encodes more information. One can recover the index of the original operator. We describe the map δ via duality, therefore we neglect the torsion of K-theory. The construction is the following: for an element $a \in K^1(M)$ we define the value of the pairing

$$\langle \delta[D], a\rangle = \langle [D], \partial a\rangle\,.$$

The element a is represented by a matrix $A \in GL_n(C^\infty(M))$. Then

$$\langle [D], \partial a\rangle = ind(D \otimes \Theta^A_X) - n\, ind(D)\,,$$

where Θ^A_X is the bundle defined in Example 4. Now

$$\langle [D], \partial a\rangle = Ind(A \bowtie H_\pm(D)) - nInd(H_\pm(D))\,.$$

5. Admissible Fredholm pairs

The following can be related to the paper of Birman and Solomyak [3] who introduced the name *admissible* for the subspaces which are the images of pseudodifferential projectors. Suppose that ξ is a vector bundle over a manifold M. We consider Fredholm pairs $H_\pm$ in $H = L^2(M;\xi)$ such that the subspaces $H_\pm$ are images of pseudodifferential projectors $P_\pm$ with symbols satisfying

$$\sigma(P_+) + \sigma(P_-) = 1\,.$$

We would like to free ourselves from the geometric context and state admissibility condition in an abstract way. We assume that H is an abstract Hilbert space with a representation of an algebra B, which is the algebra of functions on M in the geometric case. The condition that $P_\pm$ is pseudodifferential we substitute by the condition: $P_\pm$ commutes with the algebra action up to compact operators. We are ready now to give a definition:

Definition 5.1. We say that a pair of subspaces $H_\pm$ is an *admissible Fredholm pair* if there exist a pair of projectors P_ϵ for $\epsilon \in \{+,-\}$, such that $H_\epsilon = im\, P_\epsilon$ and P_ϵ commutes with the action of B up to compact operators. Moreover, we assume that $P_+ + P_- - 1$ is a compact operator.

Proposition 5.2. *Each admissible Fredholm pair is a good Fredholm pair.*

Proof. Set $K = P_+ + P_- - 1$. If $v \in H_+ \cap H_-$, then $K(v) = v$. Since K is a compact operator, $\dim(H_+ \cap H_-) < \infty$. To prove that $H_+ + H_-$ is closed and of finite codimension, note that $im(P_+ + P_-) \subset H_+ + H_-$. Since $P_+ + P_-$ is Fredholm its image is closed and of finite codimension. This way we have shown that $H_\pm$ is a Fredholm pair. Now, if we conjugate $P_+^{\oplus n}$ by ρA we obtain again an almost complementary pair of projectors. Thus $A \bowtie H_\pm$ is a Fredholm pair as well. □

We denote by $AFP(B)$ the set of good Fredholm pairs divided by the equivalence relation generated by homotopies and stabilization with respect to the direct sum. We also consider as trivial the pairs associated to projectors strictly satisfying $P_+ + P_- = 1$ and commuting with the action of B. In another words these are just direct sums of two representations of B. It is not hard to show that

Proposition 5.3.

$$AFP(B) \simeq K^1(B) \oplus \mathbf{Z}\,.$$

Proof. We have the following natural transformation:

$$\begin{array}{rccc} \beta: & AFP(M) & \to & K_1(M) \\ & (H, P_\pm) & \mapsto & (H, S_+) \end{array}.$$

Here $S_+ = 2P_+ - 1$ is s just the symmetry defined by P_+. We remind that the objects generating $K_1(M)$ are odd Fredholm modules, see [11], pp 287-289. This procedure is simply forgetting about P_-. We can recover P_- (up to homotopy) by fixing the index of the pair, i.e $\beta \oplus Ind$ is the isomorphism we are looking for. Precisely, the pseudodifferential projector is determined up to homotopy by its symbol and the index, see [9]. □

6. Splittings and polarization

We adopt the concepts of splitting and polarization to our situation.

Definition 6.1. Let H be a representation of a $\mathbf{C}^*$-algebra B in a Hilbert space. A *splitting* of H is a decomposition

$$H = H^\flat \oplus H^\sharp\,,$$

such that the projectors on the subspaces $P^\flat$, $P^\sharp$ commute with the action of B up to compact operators.

The basic example of a splitting is the one coming from a pseudodifferential projector. Another equivalent way of defining a splitting(as in [5]) is to distinguish a symmetry S, almost commuting with the action of B. Then $H^\flat$ is the eigenspace of -1 and $H^\sharp$ is the eigenspace of 1. Then we may think of H as a superspace, but we have to remember that the action of B does not preserve the grading.

Definition 6.2. In the set of splittings we introduce an equivalence relation: two splittings are equivalent if the corresponding projectors coincide up to compact operators. An equivalence class of the above relation is called a *polarization* of H.

Informally we can say, that polarization is a generalization of the symbol of a pseudodifferential projector.

Example. Let $\xi \to M$ be a complex vector bundle over a manifold. Let $\widetilde{\xi}$ be the pull back of ξ to $T^*M \setminus \{0\}$. Suppose $p : \widetilde{\xi} \to \widetilde{\xi}$ is a bundle map which is a projector (hence p is homogeneous of degree 0) . Then p defines a polarization of $L^2(M;\xi)$. Just take a pseudodifferential projector $P = P^\sharp$ with $\sigma(P) = p$ and set

$$H^\flat = ker\, P\,, \qquad H^\sharp = im\, P\,.$$

Example. Suppose (H_+, H_-) is an admissible Fredholm pair given by projectors (P_+, P_-). Then the polarizations associated with P_+ and $1-P_-$ coincide. This way an admissible Fredholm pair defines a polarization. Furthermore each polarization defines an element of $K_1(B)$.

Intuitively polarizations can be treated as a kind of orientations dividing H into the upper half and lower half. Such a tool was used in [12] to split the index of a family of Dirac operators. (In [12] splittings were called generalized spectral sections.) Polarizations were discussed in the lectures of G. Segal (see [18], Lecture 2).

7. Correspondences, bordisms, twists

Definition 7.1. We consider the category **PR** having the following objects and morphisms

- $Ob(\mathbf{PR})$ = Hilbert spaces (possibly of finite dimension) with a representation of some $\mathbf{C}^*$-algebra B and with a distinguished polarization,

- $Mor_{\mathbf{PR}}(H_1, H_2)$ = closed linear subspaces $L \subset H_1 \oplus H_2$, such that the pair $(L, H_1^\flat \oplus H_2^\sharp)$ is Fredholm.

We write also $H_1 \xrightarrow{L} H_2$.

In particular

$$Mor_{\mathbf{PR}}(H, 0) \subset Grass(H) \supset Mor_{\mathbf{PR}}(0, H)\,.$$

By Proposition 2.2 a subspace $L \subset H_1 \oplus H_2$ is a morphism if and only if

$$\Pi = P_1^\sharp \oplus P_2^\flat : L \to H_1^\sharp \oplus H_2^\flat$$

is a Fredholm operator. The composition in **PR** is the standard composition of correspondences:

$$L_1 \subset H_1 \oplus H_2\,, \qquad L_2 \subset H_2 \oplus H_3\,,$$

$$L_2 \circ L_1 = \{(x, z) \in H_1 \oplus H_3 \ : \ \exists y \in H_2\,,\ (x, y) \in L_1\,,\ (y, z) \in L_2\ \}\,.$$

In another words the morphisms are certain correspondences or relations, as they were called in [4]. Our approach also fits to the ideas of the topological field theory as presented in [18].

Proposition 7.2. *The composition of morphism is a morphism.*

Proof. Let $L_1 \in Mor_{\mathbf{PR}}(H_1, H_2)$ and $L_2 \in Mor_{\mathbf{PR}}(H_2, H_3)$. A simple linear algebra argument shows that

- the kernel of
$$\Pi_{13} : L_2 \circ L_1 \to H_1^\sharp \oplus H_3^\flat$$
is a quotient of $ker(\Pi_{12}) \oplus ker(\Pi_{23})$,
- the cokernel of Π_{13} is a subspace of $coker(\Pi_{23}) \oplus coker(\Pi_{12})$.

□

The role of polarizations in the definition of morphisms is clear and the algebra actions are involved implicitly. In fact, the object which plays the crucial role is the algebra of operators commuting with $P^\sharp$ up to compact operators, i.e. the odd universal algebra. The role of this algebra was emphasized in [5]. However, in the further presentation we prefer to expose the geometric origin of our construction and keep the name B.

We have two special classes of morphisms in **PR**:

Definition 7.3. A subspace $L \subset H \oplus H$ is a *twist* if it is the graph of a linear isomorphism $\phi \in GL(P^\sharp, K) \subset GL(H)$ commuting with the polarization projectors up to compact operators.

Proposition 7.4. *For a twist $L = graph(\phi) \subset H \oplus H$ the pair $(L, H^\flat \oplus H^\sharp)$ is Fredholm, i.e. $L \in Mor_{\mathbf{PR}}(H, H)$.*

Proof. To show that $(L, H^\flat \oplus H^\sharp)$ is a Fredholm pair let us show that the projection

$$\Pi = P^\sharp \oplus P^\flat : L \to H^\sharp \oplus H^\flat \subset H \oplus H$$

is a Fredholm operator. Indeed, L is parameterized by

$$(1, \phi) : H \to L \subset H \oplus H\,.$$

The composition of these maps is equal to

$$F = P^\sharp \oplus P^\flat \phi\,.$$

Since ϕ almost commutes with $P^\flat$ the map F has a parametrix $\widetilde{F} = P^\sharp \oplus P^\flat \phi^{-1}$. □

Definition 7.5. A subspace $L \subset H_1 \oplus H_2$ is a bordism if L is the image of a projector P_L, such that

$$P_L \sim P_1^\sharp \oplus P_2^\flat\,.$$

By 5.2 for any $P_L \sim P_1^\sharp \oplus P_2^\flat$ the pair $(L, H_1^\flat \oplus H_2^\sharp)$ is Fredholm. The motivation for the Definition 7.5 is the following:

Example. Let X be a bordism between closed manifolds M_1 and M_2, i.e.

$$\partial X = M_1 \sqcup M_2\,.$$

Suppose that $D : C^\infty(X;\xi) \to C^\infty(X;\eta)$ is an elliptic operator of the first order. Then the symbols of Calderón projectors define polarizations of $H_1 = L^2(M_1;\xi)$ and $H_2 = L^2(M_2;\xi)$, see Example 6. We reverse the polarization on M_2, i.e. we switch the roles of $H^\flat$ and $H^\sharp$. Let $L \subset L^2(M_1;\xi) \oplus L^2(M_2;\xi)$ be the closure of the space of boundary values of solutions of $Ds = 0$. Then $L \in Mor_{\mathbf{PR}}(H_1, H_2)$ is a bordism in **PR**. This procedure indicates the following:

- the space $L \subset L^2(M_1 \sqcup M_2;\xi) = L^2(M_1;\xi) \oplus L^2(M_2;\xi)$ and the associated Calderón projector are *global* objects. One cannot recover them from the separated data in $L^2(M_1;\xi)$ and $L^2(M_2;\xi)$.
- but up to compact operators one can *localize* the projector P_L and obtain two projectors acting on $L^2(M_1;\xi)$ and $L^2(M_2;\xi)$.

We note that the following proposition holds:

Proposition 7.6. 1. *The composition of bordisms is a bordism.*
2. *The composition of a bordism and a twist is a bordism.*
3. *The composition of twists is a twist.*

Remark 7.7. Let $H_1 \xrightarrow{L_1} H_2 \xrightarrow{L_2} H_3$ be a pair of bordisms in **PR** coming from geometric bordisms
$$M_1 \sim_{X_1} M_2\,, \quad M_2 \sim_{X_2} M_3$$
and an elliptic operator on $X_1 \cup_{M_2} X_2$, as in Example 7. Then $L_2 \circ L_1$ coincides with the space of the Cauchy data along $\partial(X_1 \cup_{M_2} X_2) = M_1 \sqcup M_3$ of the solutions of $Ds = 0$ on $X_1 \cup_{M_2} X_2$.

8. Chains of morphisms

Now we introduce the notion of a chain. This is a special case of a Fredholm fan considered in [5] and in §12 below.

A chain of morphisms is a sequence correspondences
$$0 \xrightarrow{L_0} H_1 \xrightarrow{L_1} H_2 \xrightarrow{L_2} \dots \xrightarrow{L_{m-1}} H_m \xrightarrow{L_m} 0\,.$$

Example. Let (H_+, H_-) be an admissible Fredholm pair in H. Then we have a sequence
$$0 \xrightarrow{H_-} H \xrightarrow{H_+} 0$$
which is a chain of bordisms with respect to the polarization defined by $P^\sharp = P_+$ (or $1 - P_-$), see Example 6.

Example. Each morphism in $L \in Mor_{\mathbf{PR}}(H_1, H_2)$ can be completed to a chain
$$0 \xrightarrow{L_1} H_1 \xrightarrow{L} H_2 \xrightarrow{L_2} 0\,.$$
Just take $L_1 = (0 \oplus H_1^\flat) \subset (0 \oplus H_1)$ and $L_2 = (H_2^\sharp \oplus 0) \subset (H_2 \oplus 0)$.

Example. It is proper to explain why we are interested in chains of morphisms. Suppose there is given a closed manifold which is composed of usual bordisms
$$X = X_0 \cup_{M_1} X_1 \cup_{M_2} \cdots \cup_{M_{m-1}} X_{m-1} \cup_{M_m} X_m\,.$$
We treat the manifolds M_i as objects and bordisms
$$M_{i-1} \sim_{X_i} M_i$$
as morphisms. In particular
$$\emptyset \sim_{X_1} M_1 \qquad \text{and} \qquad M_m \sim_{X_m} \emptyset\,.$$
Let $D : C^\infty(X; \xi) \to C^\infty(X; \eta)$ be an elliptic operator of the first order. This geometric situation gives rise to a chain of bordisms in the category **PR**:

- $H_i = L^2(M_i; \xi)$ with the action of $B_i = C(M_i)$ and the polarization defined by the symbol of Calderón projector, as in 7,
- $L_i \subset L^2(M_i; \xi) \oplus L^2(M_{i+1}; \xi)$ is the space of boundary values of the solutions of $Ds = 0$ on X_i.

9. Indices in PR

Definition 9.1. Fix the splittings S of the objects of **PR**. The pair $(L, H_1^\flat \oplus H_2^\sharp)$ in $H_1 \oplus H_2$ is Fredholm by Definition 7.1. Define the index of a morphism $L \in Mor_{\mathbf{PR}}(H_1, H_2)$ by the formula:

$$Ind_{S_1,S_2}(L) \stackrel{\text{def}}{=} Ind(L, H_1^\flat \oplus H_2^\sharp) = ind(P_1^\sharp \oplus P_2^\flat : L \to H_1^\sharp \oplus H_2^\flat)\,.$$

Proposition 9.2. *We have the equality of indices for a twist*

1. $Ind_{S,S}(graph\,\phi)$,
2. *index of* $\begin{pmatrix} 1 & P^\flat \\ \phi & P^\sharp \end{pmatrix} : H \oplus H \to H \oplus H$,
3. $\widetilde{ind}(\phi) = ind(\phi P^\flat + P^\sharp) = Ind(\phi(H^\flat), H^\sharp)$ *(compare Theorem* 2.3*),*

Proof. The graph of ϕ is parameterized by $(1, \phi)$ and $H^\flat \oplus H^\sharp$ is parameterized by $(P^\flat, P^\sharp)$. Thus by Theorem 2.1 the first equality follows. Now we multiply the matrix (2.) from the left by the symmetry $\begin{pmatrix} P^\sharp & P^\flat \\ P^\flat & P^\sharp \end{pmatrix}$ and we obtain $\begin{pmatrix} P^\flat\phi + P^\sharp & 0 \\ P^\sharp\phi + P^\flat & 1 \end{pmatrix} \sim \begin{pmatrix} \phi P^\flat + P^\sharp & 0 \\ \phi P^\sharp + P^\flat & 1 \end{pmatrix}$. The second equality follows. □

Remark 9.3. The index of a twist depends only on the polarization, not on the particular splitting. This is clear from 9.2.2. It is worthwhile to point out that if the twist $\phi = \widetilde{A} : H^{\oplus n} \to H^{\oplus n}$ is given by a matrix $A \in GL_n(B)$, then

$$\widetilde{ind}(\widetilde{A}) = \langle [\widetilde{A}], [S_{H^\flat}] \rangle\,,$$

where $S_{H^\flat}$ is the symmetry with respect to $H^\flat$ and the bracket is the pairing in K-theory of $K^1(B)$ with $K_1(B)$.

On the other hand $Ind_{S_1,S_2}(L)$ does depend on the splitting for general morphisms.

Remark 9.4. The index in Example 7 is equal to the index of the operator D with the boundary conditions given by the splittings, as in [2].

Remark 9.5. There are certain morphisms in **PR** which are interesting from the point of view of composition. We will say that L is a *special* correspondence if:

- L is the graph of an injective function ϕ defined on a subspace of H_1,
- the images of the projections of L onto H_1 and H_2 are dense.

(The second condition is equivalent to the first one for the adjoint correspondence defined as the ortogonal complement $L^\perp$.) If L is special, then

$$Ind_{S_1,S_2}(L) = Ind(L(H_1^\flat), H_2^\sharp)\,,$$

where

$$L(H_1^\flat) = \{y \in H_2 \;:\; \exists x \in H_1^\flat \;\; (x,y) \in L\,\}\,.$$

Indeed in this case we have

$$L \cap (H_1^\flat \oplus H_2^\sharp) \simeq L(H_1^\flat) \cap H_2^\sharp \quad \text{and} \quad L^\perp \cap (H_1^{\flat\,\perp} \oplus H_2^{\sharp\,\perp}) \simeq L^\perp(H_1^{\flat\,\perp}) \cap H_2^{\sharp\,\perp}\,.$$

Of course each twist is a special morphism. Another example of a special morphism is the one which comes from the Cauchy-Riemann operator. In general, we obtain a special morphism if the operator (and its adjoint) satisfies the following:

- if $s = 0$ on a hypersurface M and $Ds = 0$, then $s = 0$ on the whole component containing M.

In the set of morphisms we can introduce an equivalence relation: we say that $L \sim L'$ if L and L' are images of embeddings $i, i' : H \hookrightarrow H_1 \oplus H_2$ of a Hilbert space H, such that $i - i'$ is a compact operator. If $L \sim L'$, then $Ind_{S_1,S_2}(L) = Ind_{S_1,S_2}(L')$. If L is a bordism, then L is equivalent to a direct sum of subspaces in coordinates: $L \sim L_1 \oplus L_2$, $L_i \subset H_i$, such that L_1 is a finite dimensional perturbation of $H_1^\sharp$ and L_2 is a finite dimensional perturbation of $H_2^\flat$. Then $Ind_{S_1,S_2}(L) = Ind(H_1^\flat, L_1) + Ind(L_2, H_2^\sharp)$.

Suppose, as in Example 8, we have an elliptic operator on a closed manifold X which is composed of geometric bordisms. Fix $n \in \mathbf{N}$ and a sequence of matrices

$$A_i \in GL_n(B_i)\,.$$

Define a bundle $\Theta_X^{\{A_i\}}$ obtained from trivial ones on X_i's and twisted along M_i's. Define bordisms $L_i(D) \in Mor_{\mathbf{PR}}(H_i, H_{i+1})$ as in Example 7.

Theorem 9.6. *Suppose that* 3.1 *holds for D and D^* on each X_i for $i = 0, \ldots, n$. Then*

$$ind(D \otimes \Theta_X^{\{A_i\}}) = n\left(\sum_{i=0}^{m} Ind_{S_i,S_{i+1}}(L_i(D))\right) + \sum_{i=1}^{m} \widetilde{ind}(\widetilde{A}_i)\,.$$

Here, as it was denoted before, $\widetilde{A} : H^{\oplus n} \to H^{\oplus n}$ is the operator associated to the matrix $A \in GL_n(B)$. This Theorem is a special case of Theorem 11.1 proved below.

Taking into account Remark 9.3 the difference between the idices of the original and twisted operator can be expressed through the pairing in K-theory.

Theorem 9.7.

$$ind(D \otimes \Theta_X^{\{A_i\}}) - n\, ind(D) = \sum_{i=1}^{m} \widetilde{ind}(\widetilde{A}_i) = \sum_{i=1}^{m} \langle [A_i], [S_{H_i^\flat}] \rangle\,.$$

The braked is the pairing between $[A_i] \in K^1(M_i)$ and $[S_{H_i^\flat}] \in K_1(M_i)$.

10. Indices of compositions

In 9.3 we have made some remarks about the dependence of indices on the particular splitting. Now let us see how indices behave under compositions of correspondences. From the considerations in §9 it is easy to deduce:

Proposition 10.1. *For the composition*

$$H_1 \xrightarrow{\phi} H_1 \xrightarrow{L} H_2\,,$$

where ϕ is a twist and L is a morphism we have

$$Ind_{S_1,S_2}(L\circ\phi) = Ind_{S_1,S_2}(L) + \widetilde{ind}(\phi)\,.$$

The same holds for the opposite type composition

$$H_1 \xrightarrow{L} H_2 \xrightarrow{\phi} H_2\,,$$

$$Ind_{S_1,S_2}(\phi\circ L) = \widetilde{ind}(\phi) + Ind_{S_1,S_2}(L)\,.$$

On the other hand $Ind_{S_0,S_2}(L_2\circ L_1)$ differs from $Ind_{S_0,S_1}(L_1)+Ind_{S_1,S_2}(L_2)$ in general. This is clear due to the basic example that comes from a decomposition $X = X_- \cup_M X_+$. The space $L_1 = H_-(D)$ is a correspondence $0 \to L^2(M;\xi)$ and $L_2 = H_+(D)$ a correspondence $L^2(M;\xi) \to 0$. By 9.6 we have

$$Ind_{Id,S_1}(L_1) + Ind_{S_1,Id}(L_2) = ind(D)\,,$$

while $L_2\circ L_1 : 0 \to 0$ and $Ind_{Id,Id}(L_2\circ L_1) = 0$.

Instead we have the following interesting property of indices:

Theorem 10.2. *The difference*

$$\delta(L_1,L_2) = Ind_{S_0,S_1}(L_1) + Ind_{S_1,S_2}(L_1) - Ind_{S_0,S_2}(L_2\circ L_1)$$

does not depend on the particular splittings.

Proof. Since

$$Ind_{S_{i-1},S_i}(L_i) = ind(H^\flat_{i-1}\oplus L_i \oplus H^\sharp_i \to H_{i-1}\oplus H_i)$$

we have to compare indices of the operators

$$\alpha : H^\flat_0 \oplus L_1 \oplus H^\sharp_1 \oplus H^\flat_1 \oplus L_2 \oplus H^\sharp_2 \to H_0\oplus H_1\oplus H_1\oplus H_2$$

and

$$\beta : H^\flat_0 \oplus L_2\circ L_1 \oplus H^\sharp_2 \to H_0\oplus H_2\,.$$

The kernel of α is isomorphic to the kernel of the operator which is induced by inclusions

$$H^\flat_0 \oplus L_1 \oplus L_2 \oplus H^\sharp_2 \to H_0\oplus H_1\oplus H_2\,.$$

The former operator factors through

$$H^\flat_0 \oplus (L_1 + L_2) \oplus H^\sharp_2 \to H_0\oplus H_1\oplus H_2\,.$$

Here the direct sum is replaced by the algebraic sum inside $H_0\oplus H_1\oplus H_2$. The difference of the dimensions of the kernels is equal to the dimension of the intersection

$$(L_1\oplus 0)\cap(0\oplus L_2) \subset H_0\oplus H_1\oplus H_2$$

Now we observe that the kernel of the last operator is isomorphic to

$$H^\flat_0 \oplus L_2\circ L_1 \oplus H^\sharp_2 \to H_0\oplus H_2\,.$$

Therefore the difference of the dimensions of the kernels of α and β is equal to $\dim((L_1 \oplus 0) \cap (0 \oplus L_2))$, hence it does not depend on the splittings. We have the dual formula for cokernels and $L_i^{\perp}$, also not depending on the splittings. □

We obtain a procedure of computing the sum of indices

$$\sum_{i=0}^{m} Ind_{S_i,S_{i+1}}(L_i)$$

which would not involve splittings. We choose a pair of consecutive morphisms L_i, L_{i+1} and replace them by their compositions. The composition produces a number $\delta(L_i, L_{i+1})$ and the sequence of morphisms is shorter:

$$(L_0, L_1, \ldots, L_m) \rightsquigarrow (L_0, L_1, \ldots, L_i \circ L_{i+1}, \ldots, L_m) + \delta(L_i, L_{i+1})\,.$$

We pick another composition and add its contribution to the previous one. We continue until we get $0 \to 0$. The sum of the contributions does not depend on the splittings. One can perform compositions in various ways. The sum of contributions stays the same.

Example. If D and D^* on X_i and X_{i+1} have the unique extension property 3.1, then $\delta(L_i, L_{i+1}) = 0$ as long the gluing process along M_{i+1} does not create a closed component of X. If it does then $\delta(L_i, L_{i+1})$ equals to the index of D restricted to this component.

11. Weird decompositions of manifolds

Let $\{M_e\}_{e\in E}$ be a configuration of disjoined hypersurfaces in a manifold X. We assume that orientations of the normal bundles are fixed. For simplicity assume that X and M_e's are connected. Let

$$X \setminus \bigsqcup_{e\in E} M_e = \bigsqcup_{v\in V} X_v$$

be the decomposition of X into connected components. Our situation is well described by an oriented graph

- the vertices (corresponding to open domains in X) are labelled by the set V
- the edges (corresponding to hypersurfaces) are labelled by E. The edge e starts at the vertex $v = s(e)$ corresponding to X_v which is on the negative side of M_e. It ends at $v' = t(e)$, such that $X_{v'}$ lies on the positive side of M_e. The functions $s, t : E \to V$ are the *source* and *target* functions.

For example the configuration of the cutting circles on the surface (Fig. 1)

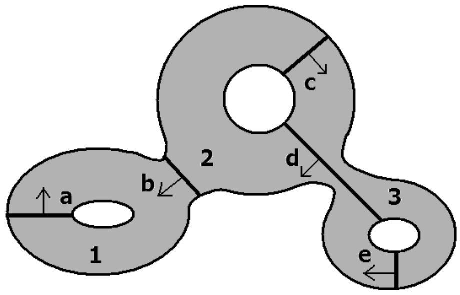

Fig. 1

is described by the graph (Fig. 2):

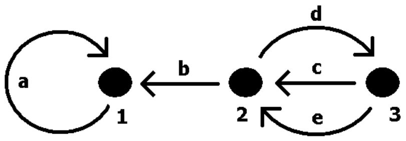

Fig. 2

A sequence of bordisms leads to the linear graph

$$\bullet_{X_0} \xrightarrow{M_1} \bullet_{X_1} \xrightarrow{M_2} \dots \xrightarrow{M_{n-1}} \bullet_{X_{n-1}} \xrightarrow{M_n} \bullet_{X_n} \ .$$

Note that this is a dual description with respect to the one presented in Example 8. Suppose there is given an elliptic operator $D : C^\infty(X;\xi) \to C^\infty(X;\eta)$ and a set of transmission data $\{\phi_e\}_{e\in E}$, that is for each hypersurface M_e we are given a matrix-valued function $M_e \to GL_n(\mathbf{C})$. The Riemann-Hilbert problem gives rise to the operator

$$D^{[\phi]} : \bigoplus_{v\in V} C^\infty(X_v;\xi)^n \to \bigoplus_{v\in V} C^\infty(X_v;\eta)^n \oplus \bigoplus_{e\in E} C^\infty(M_e;\xi)^n$$

$$D^{[\phi]}(f_v) \stackrel{\text{def}}{=} \left(Df_v, \sum_{e:\, t(e)=v} f_{v|M_e} - \sum_{e:\, s(e)=v} \phi_e(f_{v|M_e}) \right) , \quad \text{for} \quad f_v \in C^\infty(X_v;\xi)^n \, .$$

For $e \in E$ let us set $H(e) = L^2(M_e;\xi)$. The symbol of D together with the choice of orientations of the normal bundles define polarizations of $H(e)$. Let us fix particular splittings of the spaces $H(e)$ encoded in the symetries S_e. Set

$$\begin{aligned} H^{\text{bd}}(v) &= \bigoplus_{e:\, s(e)=v} H(e) \oplus \bigoplus_{e:\, t(e)=v} H(e)\,, \\ H^{\text{in}}(v) &= \bigoplus_{e:\, s(e)=v} H^\sharp(e) \oplus \bigoplus_{e:\, t(e)=v} H^\flat(e)\,, \\ H^{\text{out}}(v) &= \bigoplus_{e:\, s(e)=v} H^\flat(e) \oplus \bigoplus_{e:\, t(e)=v} H^\sharp(e)\,. \end{aligned}$$

Let $L(v) \subset H^{\text{bd}}(v)$ be the space of boundary values of solutions of $Df_v = 0$ on X_v. It is a perturbation of $H^{\text{in}}(v)$. For each vertex v (i.e. for each open domain

X_v) the pair of subspaces

$$L(v), H^{\rm out}(v) \subset H^{\rm bd}(v)\,,$$

is Fredholm. Let Ind_v be its index with respect to the polarizations S_e. Moreover, let $Ind_e = Ind_{S_e,S_e}(\phi_e) = \widetilde{ind}(\phi_e)$ denote the index of ϕ_e, see Theorem 2.3.

Theorem 11.1. *Assume that D and D^* have unique extension property (3.1) on each X_v. Then*

$$ind(D^{[\phi]}) = \sum_{v\in V} Ind_v + \sum_{e\in E} Ind_e\,.$$

In particular:

Corollary 11.2. *If there are no twists, i.e. each $\phi_e = 1 \in GL_1(C^\infty(M_e))$, then*

$$ind(D) = \sum_{v\in V} Ind_v\,.$$

Proof. (of 11.1.) The general result follows from the case when we have one vertex and one edge starting and ending in it. We just sum up all X_v's and all M_e's. Say that X is obtained from $\hat{X}$ with $\partial\hat{X} = M_s \sqcup M_t$ by identification M_s with M_t as presented on Fig. 3.

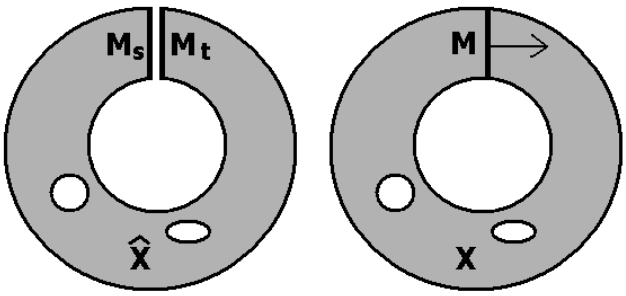

Fig. 3

Then our operator $D^{[\phi]}$ is of the form:

$$D^{[\phi]} : C^\infty(\hat{X};\xi)^n \to C^\infty(\hat{X};\eta)^n \oplus C^\infty(M;\xi)^n$$

$$D^{[\phi]}(u) = \big(Du, u_{|M_t} - \phi(u_{|M_s})\big)\,.$$

We replace $\xi^{\oplus n}$ by ξ and treat ϕ as an automorphism of ξ. The index of the operator is equal to the index of a Fredholm pair:

Theorem 11.3. *Let $L \subset L^2(M_s \sqcup M_t;\xi) = L^2(M;\xi) \times L^2(M;\xi)$ be the space of boundary values of the operator D on $\hat{X}$. Then*

$$ind(D^{[\phi]}) = Ind(L, graph(\phi))\,.$$

The proof of Theorem 11.1 relies on this formula. We will give a heuristic proof of 11.3. The precise argument demands introduction and consecutive use of the whole scale of Sobolev spaces with all usual technicalities involved. The reader may also take this formula as the definition of the index of the problem considered above. We calculate the kernel and cokernel of $D^{[\phi]}$:

- the kernel consist of solutions of $Du = 0$ on $\hat{X}$ satisfying $\phi(u_{|M_s}) = u_{|M_t}$. By our assumption u is determined by its boundary value. Thus
$$ker\, D^{[\phi]} \simeq L \cap graph\, \phi\,.$$

The cokernel consists of

$$\Big\{(v,w) \in C^\infty(\hat{X};\eta^*) \oplus C^\infty(M;\xi^*) : \\ \forall u \in C^\infty(X_+;\xi) \quad \langle Du, v\rangle + \langle u_{|M_t} - \phi(u_{|M_s}), w\rangle = 0\Big\}\ .$$

Let $G : \xi_{|M} \to \eta_{|M}$ be the isomorphism of the bundles defined by the symbol of D as in [14]. It follows that

- $D^*v = 0$ (since we can take any u with support in $int\,\hat{X}$)
- by Green formula $\langle Du, v\rangle = \langle Gu_{|M_s}, v_{|M_s}\rangle + \langle Gu_{|M_t}, v_{|M_t}\rangle$
- since $u_{|M_s}$ and $u_{|M_t}$ may be arbitrary it follows that
  $G^*(v_{|M_s}) = -\phi^* w$,
  $G^*(v_{|M_t}) = w$,
- therefore $v_{|M_s} = -G^{*-1}\phi^* G^*(v_{|M_t})$.

Now we use the identification

$$G^* \times G^* : L^2(M_s;\eta^*) \times L^2(M_t;\eta^*) \to L^2(M_s;\xi^*) \times L^2(M_t;\xi^*)$$

under which $L^\perp$ is equal to the space of boundary values $H(D^*)$ and

$$(graph\,\phi)^\perp = (graph(-G^{*-1}\phi^* G^*))^{op}\,.$$

(Here the opposite correspondence R^{op} is defined by $(x,y) \in R^{op} \equiv (y,x) \in R$.) In another words ϕ and $G^{*-1}\phi^* G^*$ are adjoined. Since the boundary values of v determine v we can identify

$$coker\, D^{[\phi]} \simeq H(D^*) \cap (-graph(G^{*-1}\phi^* G^*))^{op} \simeq L^\perp \cap (graph\,\phi)^\perp\,.$$

□

Proof. (Continuation of 11.1.) After fixing a splitting of $L^2(M;\xi) = H_e$, we have in our notation $H_v^{\rm in} = H^\flat \oplus H^\sharp$, $H_v^{\rm out} = H^\sharp \oplus H^\flat$. By 2.3 there exists an linear isomorphism $\Psi : H \oplus H \to H \oplus H$ almost commuting with $P^\flat \oplus P^\sharp$, such that $L = \Psi(H^\flat \oplus H^\sharp)$. We parameterize the graph of ϕ by $H^\sharp \oplus H^\flat$ using the composition $\Phi = \begin{pmatrix} 1 & 0 \\ \phi & 1\end{pmatrix} \circ \begin{pmatrix} P^\sharp & P^\flat \\ P^\flat & P^\sharp\end{pmatrix}$. Thus

$$Ind(graph\,\phi, L) = ind\left(\Phi \circ \begin{pmatrix} P^\sharp & 0 \\ 0 & P^\flat\end{pmatrix} + \Psi \circ \begin{pmatrix} P^\flat & 0 \\ 0 & P^\sharp\end{pmatrix}\right)\,.$$

Since Ψ almost commutes with $P^\flat \oplus P^\sharp$, the considered operator is almost equal to the composition

$$\left(\Phi \circ \begin{pmatrix} P^\sharp & 0 \\ 0 & P^\flat\end{pmatrix} + \begin{pmatrix} P^\flat & 0 \\ 0 & P^\sharp\end{pmatrix}\right) \circ \left(\begin{pmatrix} P^\sharp & 0 \\ 0 & P^\flat\end{pmatrix} + \Psi \circ \begin{pmatrix} P^\flat & 0 \\ 0 & P^\sharp\end{pmatrix}\right)\,.$$

Now we use additivity of indices. The index of the second term is equal to Ind_v. It remains to compute the first index, that is $ind\begin{pmatrix} 1 & P^\flat \\ \phi P^\sharp & \phi P^\flat + P^\sharp \end{pmatrix}$. If we conjugate the above matrix by the symmetry $\begin{pmatrix} P^\sharp & P^\flat \\ P^\flat & P^\sharp \end{pmatrix}$ we obtain $\begin{pmatrix} P^\sharp + P^\flat\phi & 0 \\ P^\flat + P^\sharp\phi & 1 \end{pmatrix}$. Its index is equal to $ind(P^\sharp + P^\flat\phi) = Ind_e$. □

The additivity of the index is not a surprise due to the well known integral formula for the analytic index. What is interesting in Theorem 11.2 is that the contribution coming from separate pieces of X is also an integer number. This partition into local indices depends only on the choice of splittings along hypersurfaces.

12. Index of a fan

We will give another formula for the index of $D^{[\phi]}$ which is expressed in terms of the twisted fan $\{L(i)\}$. The general reference for fans is [5]. Let us first say what we mean by a fan: it is a collection of spaces

$$L_1, L_2, \dots, L_n \subset H$$

which is obtained from a direct sum decomposition

$$H_1 \oplus H_2 \oplus \dots \oplus H_n = H$$

by a sequence of twists $\Psi_1, \Psi_2, \dots, \Psi_n \in GL(H)$, i.e. $L_i = \Psi_i(H_i)$. We assume that each Ψ_i almost commutes with each projection P_j of the direct sum. We say that the fan $\{L(i)\}$ is a perturbation of the direct sum decomposition $H = \oplus H_i$.

Theorem 12.1 (Index of a Fredholm fan). *Let $L_1, L_2, \dots, L_n \subset H$ be a fan. Then the following numbers are equal:*

1. *the index of the map $\iota : L_1 \oplus L_2 \oplus \dots \oplus L_n \to H$, which is the sum of inclusions,*
2. *the index of the operator $\Psi_1 P_1 + \Psi_2 P_2 + \dots + \Psi_n P_n : H \to H$,*
3. *the sum*
$$\sum_{i=1}^{n} ind(P_i\Psi_i : H_i \to H_i) = \sum_{i=1}^{n} ind(P_i : L_i \to H_i)\,,$$
4. *the difference*
$$\sum_{i=1}^{n-1} \dim(L_1 + \dots + L_i) \cap L_{i+1} - \operatorname{codim}(L_1 + \dots + L_n)\,.$$

Proof. The equality (1.=2.) follows from the fact that $\Psi_i : H_i \to P_i$ is a parameterization of L_i. The equality (2.=3.) follows since

$$\Psi_1 P_1 + \Psi_2 P_2 + \dots + \Psi_n P_n \sim \prod_{i=1}^{n} (P_1 + \dots + \Psi_i P_i + \dots + P_n)\,.$$

To prove the equality (1.=4.) one checks that

$$\dim(ker\,\iota) = \sum_{i=1}^{n-1} (L_1 + \cdots + L_i) \cap L_{i+1}\,.$$

This is done by induction with respect to n. □

Let us assume that the graph associated to our configuration does not contain edges starting and ending in the same vertex (e.g. the situation on fig.1 is not allowed). Then $H^{\rm bd}(v)$ is a summand in $H = \bigoplus_{e\in E} H(e)$ (there are no terms $H(e)$ appearing twice). Moreover, $\{L(v)\}_{v\in V}$ is a fan in H which is a perturbation of the direct sum decomposition

$$H = \bigoplus_{v\in V} H^{\rm in}(v)\,.$$

Consider a fan, which is twisted with respect to $\{L(v)\}_{v\in V}$. Set $(\phi\bowtie L)(v) = \widetilde{\phi}_v(L(v))$, where $\widetilde{\phi}_v$ is an automorphisms of H:

$$\widetilde{\phi}_v(f) \stackrel{\rm def}{=} \begin{cases} \phi_e(f) & \text{if } f\in H(e),\ s(e)=v\,, \\ f & \text{if } f\in H(e),\ s(e)\neq v\,. \end{cases}$$

Theorem 12.2. *Assume that D and D^* have unique extension property (3.1) on each X_v. The index of $D^{[\phi]}$ is equal to the index of the Fredholm fan $\phi\bowtie L$.*

Proof. Combining Theorem 11.1 with 12.1.3 it remains to prove that for each vertex v

$$ind(P^{\rm in}_v : (\phi\bowtie L)(v) \to H^{\rm in}(v)) = Ind_v + \sum_{e\,:\,s(e)=v} Ind_e\,.$$

If there are no twists, then the equality follows from Proposition 2.2. In general the proof follows from additivity of $\widetilde{ind}$, see Theorem 2.3. □

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