LERAY RESIDUE FORM AND INTERSECTION HOMOLOGY

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Abstract. We consider a meromorphic form with a first order pole along a hypersurface K. We ask when the Leray residue form determines an element in intersection homology of K. We concentrate on K with isolated singularities. We find that the mixed Hodge structure on vanishing cycles plays a decisive role. We give various conditions on the singularities of K which guaranties that residues lie in intersection homology. For $\dim K > 1$ all simple singularities satisfy these conditions.

Contents:

0. Introduction

Let M be a complex manifold of dimension $n+1$ and let K be a smooth hypersurface. Let $Tub\,K$ be a tubular neighbourhood of K. Let us consider a commutative diagram:

$$\begin{array}{ccccc} H^*(M\setminus K) & \xrightarrow{\ \delta\ } & H^{*+1}(M,M\setminus K) & = & H^{*+1}(Tub\,K,Tub\,K\setminus K)\\ & & \big\downarrow{\scriptstyle\cap[M]} & & \big\uparrow{\scriptstyle\tau}\\ & & H^{BM}_{2n+1-*}(K) & \xleftarrow{\ \cap[K]\ } & H^{*-1}(K)\,. \end{array}$$

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In the diagram H_*^{BM} denotes Borel–Moore homology, i.e. homology with closed supports. All coefficients are in $\mathbb{C}$. The map τ is the Thom isomorphism, the remaining maps in the square are also isomorphisms by Poincaré duality for K and M. The residue map

$$res = \tau^{-1} \circ \delta : H^*(M \setminus K) \longrightarrow H^{*-1}(K)$$

is defined to be the composition of the differential with the inverse of the Thom isomorphism.

Now suppose that K is singular. Then there is no tubular neighbourhood of K nor Thom isomorphism, but we can still define a residue morphism

$$res : H^*(M \setminus K) \longrightarrow H^{BM}_{2n+1-*}(K)$$

$$res\, c = \delta c \cap [M]$$

If K was nonsingular, then this definition would be equivalent to the previous one since $\xi \mapsto \xi \cap [K]$ is Poincaré duality isomorphism and the diagram above commutes. In general there is no hope to lift the residue morphism to cohomology. For $M = \mathbb{C}^{n+1}$ the morphism res is the Alexander duality isomorphism and $\cap[K]$ may be not onto. Instead we ask if the residue of an element lifts to the intersection homology of K. The intersection homology groups, defined by Goresky and MacPherson [GM], are the functors that 'lie between' homology and cohomology; i.e. there is a factorization:

$$\begin{array}{ccc} H^*(K) & \xrightarrow{\cap[K]} & H^{BM}_{2n-*}(K) \\ & \searrow \quad \nearrow & \\ & IH^{p}_{2n-*}(K)\,. & \end{array}$$

For K with isolated singularities the intersection homology is just homology or cohomology or the image of the Poincaré morphism.

Let ω be a closed form with a first order pole on K. Then the residue form $Res\,\omega$ can be defined at the regular points of K. (We use the capital letter for $Res\,\omega \in \Omega^*_{K_{reg}}$ to distinguish it from $res\,\omega \in H^{BM}_{2n+1-*}(K)$. Mostly we discuss the case when ω is a holomorphic $n+1$–form:

$$\omega = \frac{g}{s} dz_0 \wedge \dots \wedge dz_n\,,$$

where the function s describes K. The space of such forms is denoted by $\mathcal{O}^{(n+1)}_M(K)$. Then the residue form is a holomorphic n–form:

$$Res\,\omega \in \mathcal{O}^{(n)}_{K_{reg}}\,.$$

The purpose of this paper is to give few conditions which guarantee that the residue form defines an element in intersection homology provided that K has isolated singularities.

The paper is organized as follows. In §1 recall the construction of Leray residue form. Next we describe intersection homology for K with isolated singularities. For the middle perversity $\underline{m}$ and the middle dimension n we have

$$IH^{\underline{m}}_n(K) = im\left(H^n(K) \xrightarrow{\cap[K]} H_n(K)\right)\,.$$

We neglect the case $n = 1$. The residue form determines an element in $IH^{\frac{m}{n}}(K)$ if and only if it vanishes in cohomology when restricted to the link of each singular point.

In the paragraph 2 we describe the topological structure of a neighbourhood of a singular point. We stress the importance of the Milnor fibration and the monodromy. This is also a source of our examples. We recall the forms of simple and unimodal parabolic singularities.

In §3 we discuss the mixed Hodge structure on vanishing cycles and the notion of the spectrum of an isolated singularity. Spectrum is a set of rationals associated with a singular point. It arises from a set of possible exponents in oscillating integrals of Arnold and Varchenko. The following theorem shows its importance for our problem:

Theorem 0.1. *If each residue class lifts to the intersection homology of K, then the number 0 does not belong to the spectra of the singular points of K.*

In §§4-6 and §8 we consider K with quasihomogeneous singularities. Then the converse of 0.1 holds. Suppose a function s describing K in local coordinates is quasihomogeneous of degree 1 with respect to weights a_0, a_1, $\ldots$ a_n. Let $\kappa = a_0 + a_1 + \cdots + a_n$. In §4 we formulate a condition which is equivalent to the one in 0.1:

Condition 0.2. *For any choice of nonnegative integers $k_i \in \mathbb{N} \cup \{0\}$, $i = 0, \ldots, n$ we have $\kappa + \sum k_i a_i \neq 1$.*

In the next paragraph we prove without use of the theory of oscillating integrals that 0.3 implies the existence of a lift. In §6 we show that in fact 0.3 is a necessary condition. We discover an obstruction to lift, 'second residue', which does not vanish for the forms of weight 1.

Next we start to investigate more concrete method of lift. In §7 we recall the isomorphism of intersection homology and L^p-cohomology. Our goal is to find a conelike metric in neighbourhoods of singular points for which the residue form would have its norm in L^p. In §8 we prove the following:

Theorem 0.3. *Let s be a polynomial in $n+1$ variables. Suppose it is quasihomogeneous of the weight 1 with respect to weights $a_0, \ldots, a_n$. Assume that 0 is the only critical point of s. If $\kappa > 1$ then there exists a conelike metric on $K = \{s = 0\}$ such that the norms of the residue forms $Res\,\omega$ are in $L^p(K)$.*

This gives a lift of residue class to intersection homology provided that $\kappa > 1$ at each singular point. Note that this condition clearly implies 0.3.

The purpose of §9 is to set our problem in a context of $\mathcal{D}$-modules. Recall the sheaf ($\mathcal{D}$-module) of the meromorphic parts of functions with poles on K: $\mathcal{O}_M(*K)/\mathcal{O}_M = \mathcal{H}^1_K(\mathcal{O}_M)$. It contains an unique irreducible submodule $\mathcal{L}(K)$. The sheaf $\mathcal{L}(K)$ corresponds to intersection homology sheaf by Riemann-Hilbert correspondence. We observe that:

Proposition 0.4. *If $\mathcal{L}(K)$ contains all the function with the first order pole on K then every residue form defines an element in intersection homology.*

This proposition remains true for arbitrary singularities (possibly nonisolated). The uniqueness of the lift in the derived category of sheaves is proved in §10.

When M is algebraic, then its cohomology is equipped with a mixed Hodge structure. Paragraph 11 indicates that our problem is strictly connected with it. We wish to explore this direction in future. We also do not discuss a relation of our problem with problem of lifting singular forms to a resolution.

In the Appendix we give an example of P_8 singularity, for which $\kappa = 1$. The second residue has very interesting form: we obtain an elliptic integral.

I was involved in investigating multidimensional residues by Professor Bogdan Ziemian (see [Zi]) and I would like to thank him first of all. I would also like to thank Professors G. Barthel, J-P. Brasselet, P. Jaworski, Z. Marciniak, K. Vilonen and H. Żołądek for help and valuable comments. The paper [We3] announces the results obtained here and contains some examples for K with nonisolated singularities.

1. Residue forms and intersection homology

We recall the Leray method of defining the residue form [Le]. Let ω be a smooth closed k–form on the complement of the set

$$K = \{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} : s(z_0, \ldots, z_n) = 0\},$$

where s is holomorphic. Suppose that ω has a first order pole on K; i.e. $s\,\omega$ is a global form on $\mathbb{C}^n$. At the points where $ds \neq 0$ the form ω can be written locally as

$$\omega = \frac{ds}{s} \wedge \boldsymbol{r} + \eta\,, \tag{1.1}$$

where $\boldsymbol{r}$ and η have no pole on K. Let $\Sigma_K = K \cap \{ds = 0\}$ be the singular set of K and $K_{reg} = K \setminus \Sigma_K$ be the set of regular points. The form

$$\boldsymbol{r}_{|K} \in \Omega^{k-1}_{K_{reg}}$$

does not depend on the presentation 1.1 and on the function s describing K. It is called the residue form of ω and denoted by $Res\,\omega$. Thus it is defined globally for a hypersurface in a complex manifold. Moreover, $Res\,\omega$ is closed on K and its class in $H^{k-1}(K_{reg})$ does not depend on the representative of the class $[\omega] \in H^k(M \setminus K)$. For a smooth K the form $2\pi i\,Res\,\omega$ represents the class $res\,\omega$. It represents the residue class $2\pi\,i\,res\,\omega$, where $res\,\omega$ is the class defined in the introduction by cohomological methods; [Do], [Le], [SS].

We are particularly interested in holomorphic forms of degree $(n+1, 0)$. Let ω be such a form. Locally it can be written as

$$\omega = \frac{g}{s} dz_0 \wedge \cdots \wedge dz_n \in \mathcal{O}^{(n+1)}_M(K)$$

with g holomorphic. Set $s_i = \frac{\partial s}{\partial z_i}$. We have

$$ds = \sum_{i=0}^{n} s_i dz_i\,.$$

At the points where $s_0 \neq 0$ we write

$$dz_0 = s_0^{-1}\left(ds - \sum_{i=1}^{n} s_i dz_i\right)$$

and

$$\omega = \frac{g}{s\,s_0}\left(ds - \sum_{i=1}^{n} s_i dz_i\right) \wedge dz_1 \wedge \cdots \wedge dz_n =$$

$$= \frac{ds}{s} \wedge \frac{g}{s_0} dz_1 \wedge \cdots \wedge dz_n\,.$$

Thus $Res\,\omega = \left(\frac{g}{s_0} dz_1 \wedge \cdots \wedge dz_n\right)_{|K_{reg}} \in \mathcal{O}^{(n)}_{K_{reg}}$.

To see how $Res\,\omega$ behaves in a neighbourhood of the singularities let us calculate its norm in the metric coming from the coordinate system:

$$|Res\,\omega|_K = \left|\frac{ds}{|ds|} \wedge \boldsymbol{r}\right| = \left|\frac{s\,\omega}{|ds|}\right| = \frac{|g|}{|grad\,s|}\,.$$

We conclude that $Res\,\omega$ has (in general) a pole at singular points of K.

The forms that can appear as residue forms are exactly the regular differential forms defined by Kunz for arbitrary varieties; [Ku].

Suppose K has isolated singularities. Define K° to be K minus the sum of small open balls centered at the singular points of K. Let $j : (K^\circ, \emptyset) \longrightarrow (K^\circ, \partial K^\circ)$ and $k : \partial K^\circ \longrightarrow K^\circ$ be the inclusions. Assume $\dim K = n > 1$. By Poincaré duality for $K^\circ \simeq K_{reg}$ we have:

$$H^{BM}_n(K) \simeq H^{BM}_n(K^\circ, \partial K^\circ) \simeq H^n(K^\circ) \simeq H^n(K_{reg})$$

and $2\pi i[Res\,\omega] \in H^n(K_{reg})$ again coincides with (this time) homology residue class $res\,\omega = \delta\omega \cap [M]$. The intersection homology (for the middle perversity and closed supports) is in this case [Bo §5.1]:

$$\begin{aligned} IH^{\underline{m}}_n(K) &= im\left(j_* : H^{BM}_n(K^\circ) \longrightarrow H^{BM}_n(K^\circ, \partial K^\circ)\right) = \\ &= im\,(j^* : H^n(K^\circ, \partial K^\circ) \longrightarrow H^n(K^\circ)) = \\ &= ker\,(k^* : H^n(K^\circ) \longrightarrow H^n(\partial K^\circ)) \subset H^n(K^\circ)\,. \end{aligned}$$

The canonical morphism $IH^{\underline{m}}_n(K) \longrightarrow H^{BM}_n(K) \simeq H^n(K^\circ)$ is just the inclusion of $ker\,k^*$. It coincides with the inclusion of the image of the Poincaré duality map

$$PD : H^k(K) \xrightarrow{\cap[K]} H^{BM}_{2n-k}(K)\,.$$

We see that:

Proposition 1.2. *If K has isolated singularities,* $\dim K = n > 1$ *then $IH_n(K)$ coincides with the set of those classes $[\eta] \in H^n(K^\circ)$ for which*

$$\int_\zeta \eta = 0$$

for all cycles ζ in $S_\epsilon \cap K$, where S_ϵ is small sphere centered in a singular point.

The intersection homology groups in the remaining dimensions are

$$IH^{\underline{m}}_k(K) \simeq \begin{cases} H^{BM}_k(K) \simeq H^{2n-k}(K^\circ) & \text{for} \quad k > n \\ H^{2n-k}(K) \simeq H^{2n-k}(K^\circ, \partial K^\circ) & \text{for} \quad k < n\,. \end{cases}$$

There is another description of $IH^{\underline{m}}_*(K)$ which is considered in §7. It consists of the classes which can be represented by forms with square integrable norms (in a suitable metric). Our goal in §8 will be to check whether $|Res\,\omega|$ is square integrable.

Now let us present examples.

Example 1.3. Let $s = xy$ and let $\omega = \frac{1}{s} dx \wedge dy$. Then $ds = y\,dx + x\,dy$. The residue form is $\frac{dy}{y}$ for $x = 0$ and $\frac{-dx}{x}$ for $y = 0$. Let $D_\epsilon = \{z \in \mathbb{C} : |z| < \epsilon\}$. We see that

$$K^\circ = (\mathbb{C} \setminus D_\epsilon) \times \{0\} \cup \{0\} \times (\mathbb{C} \setminus D_\epsilon)$$

and $Res\,\omega$ does not belong to $ker\,k^* = IH_1^{\overline{m}}(K) = \{0\}$ since the form $\frac{dy}{y}$ (and $\frac{-dx}{x}$) is a generator when restricted to a circle around 0.

Since one may think, that the example 1.3 is degenerate (K is not normal and $\dim K = 1$) let us consider another one.

Example 1.4. Let s be a singularity of the type P_8:

$$s(z_0, z_1, z_2) = z_0^3 + z_1^3 + z_2^3\,.$$

The residue class is $Res(\frac{1}{s} dz_0 \wedge dz_1 \wedge dz_2) = \frac{1}{3z_0^2} dz_1 \wedge dz_2$ for $z_0 \neq 0$. It has no lift to intersection homology; see 6.5.

2. Topology of a neighbourhood of a singular point

Let us assume that $0 \in \mathbb{C}^{n+1}$ is an isolated singular point of a hypersurface K. Intersect K with a sphere of a small radius. Then the set $L = S_\epsilon \cap K$ is called the link of the singular point. Milnor [Mi] gave the precise description of the topology of L. It is $2n-1$ dimensional manifold with nonzero homology only in dimensions 0, $n-1$, n and $2n-1$. Consider the Milnor fibration

$$\bar{s} : s^{-1}(\dot{D}_\delta) \cap B_\epsilon \longrightarrow \dot{D}_\delta\,,$$

where $\dot{D}_\delta$ is the punctured disc in $\mathbb{C}$ of the radius δ which is much smaller then a sufficiently small ϵ. Let $K_t = \bar{s}^{-1}(t)$ for $t \in \dot{D}_\delta$. The boundary $\partial K_t = s^{-1}(t) \cap S_\epsilon$ is homeomorphic to L. Let h_* be the monodromy acting on the homology of the Milnor fiber $H_n(K_t)$ and let $\Delta(t)$ be its characteristic polynomial.

Theorem 2.1. [Mi, 8.5]. *Let $n \geq 1$. The link of an isolated singular point of $s : \mathbb{C}^{n+1} \to \mathbb{C}$ is a rational homology sphere if and only if $\Delta(1) \neq 0$, i.e. 1 is not a eigenvalue of the monodromy. If $n \neq 2$ then the link is homeomorphic to a sphere if and only with $\Delta(1) = \pm 1$.*

We will make Theorem 2.1 more precise. Denote by $H_n(K_t)^{h_*}$ the invariant cycles under the action of the monodromy.

Proposition 2.2. *There is an isomorphism $H_n(K_t)^{h_*} \simeq H_n(L)$ for $n > 1$.*

Proof. By the Thom isomorphism $H_n(L) \simeq H_{n+2}(S_\epsilon, S_\epsilon \setminus L)$ and the second term is isomorphic to $H_{n+1}(S_\epsilon \setminus L)$. The space $S_\epsilon \setminus L$ is homeomorphic to the space of the Milnor fibration restricted to a circle. By the Wang sequence we obtain the thesis. □

Remark. When we trace the geometry hidden behind the maps we recover that the isomorphism is induced by the identification $L \simeq \partial K_t$ and the inclusion $\partial K_t \subset K_t$.

Milnor described a recipe for computing the characteristic polynomial $\Delta(t)$ of a quasihomogeneous function. We restrict our attention to the case of the simple and the unimodal parabolic (simply elliptic) singularities, [AGVI]. All these types may be represented by quasihomogeneous polynomials. Our choice is motivated by the fact that every singularity is simple (i.e. it is of the type: A_k, D_k, E_6, E_7, E_8) or it is adjacent to one of the unimodal parabolic type (i.e. to P_8, X_9 or J_{10}). We list the families of simple singularities and the corresponding characteristic polynomials. The table contains answers to the following questions:

a) Is the link homeomorphic to a sphere?
b) Is it a rational sphere?

Singularity type	k	n	characteristic polynomial	a)	b)
$A_k : z_0^{k+1} + \sum_{i=1}^n z_i^2$	odd	odd	$\pm(t^k - t^{k-1} + \cdots \pm 1)$	no	no
	even	odd		yes	yes
	all	even	$t^k + t^{k-1} + \cdots + 1$	no	yes
$D_k : z_0^2 z_1 + z_1^{k-1} + \sum_{i=2}^n z_i^2$	≥ 4	odd	$\pm(t-1)(t^{k-1} - (-1)t^k)$	no	no
	≥ 4	even	$\pm(t+1)(t^{k-1} + 1)$	no	yes
$E_6 : z_0^3 + z_1^4 + \sum_{i=2}^n z_i^2$		odd	$t^6 - t^5 + t^3 - t + 1$	yes	yes
		even	$t^6 + t^5 - t^3 + t + 1$	no	yes
$E_7 : z_0^3 + z_0 z_1^3 + \sum_{i=2}^n z_i^2$		odd	$-(t-1)(t^6 + t^3 + 1)$	no	no
		even	$-(t+1)(t^6 - t^3 + 1)$	no	yes
$E_8 : z_0^3 + z_1^5 + \sum_{i=2}^n z_i^2$		odd	$t^8 - t^7 + t^5 - t^4 + t^3 - t + 1$	yes	yes
		even	$t^8 + t^7 - t^5 - t^4 - t^3 + t + 1$	yes	yes

The unimodal parabolic singularities are as follows[1]:

Singularity type	n	characteristic polynomial	a)	b)
$P_8 : z_0^3 + z_1^3 + z_2^3 + a z_1 z_2 z_3 + \sum_{i=3}^n z_i^2$	odd	$(t^3+1)^2(t^2 - t + 1)$	no	yes
	even	$(t^3-1)^2(t^2 + t + 1)$	no	no
$X_9 : z_0^4 + z_1^4 + a z_2^2 z_2^2 + \sum_{i=2}^n z_i^2$	odd	$-(t^4-1)^2(t-1)$	no	no
	even	$-(t^4-1)^2(t+1)$	no	no
$J_{10} : z_0^3 + z_1^6 + a z_0^2 z_1^2 + \sum_{i=2}^n z_i^2$	odd	$(t^6-1)(t^3+1)(t-1)$	no	no
	odd	$(t^6-1)(t^3-1)(t+1)$	no	no

We see that the link of a singular point often happens to be a rational homology sphere. If it is the case then $K = \{s = 0\}$ is a rational homology manifold and the Poincaré duality map

$$PD : H^k(K) \xrightarrow{\cap[K]} H^{BM}_{2n-k}(K)$$

is an isomorphism. Thus each residue class lifts to cohomology. For other cases there it is not possible to construct a lift of the residue morphism.

3. Spectrum of an isolated singularity

Recall few elements of the theory of oscillating integrals. The general reference to this paragraph is [AGVII] where the reader can find a review of the whole theory, a sample of proofs and precise references to original papers. We warn that in [AGVII] authors consider singularities of functions of n variables, thus citing formulas we put $n+1$ instead of n.

Suppose $0 \in \mathbb{C}^{n+1}$ is an isolated singular point of s. There is given a germ at 0 of a holomorphic $(n+1)$-form $\eta \in \mathcal{O}^{(n+1)}_{\mathbb{C}^{n+1},0}$. Define a quotient of forms by:

$$(\eta/ds)_{|s^{-1}(t)} = Res\left(\frac{\eta}{s-t}\right).$$

[1]The number a is such that: $a^3 + 27 \neq 0$ for P_8, $a^2 \neq 4$ for X_9 and $4a^3 + 27 \neq 0$ for J_{10}

Let $\zeta_t \subset K_t$, $t \in \dot{D}_\delta$ be a continuous multivalued family of n–cycles in the Milnor fibres. The function

$$I^\omega_\zeta(t) = \int_{\zeta_t} \eta/ds$$

is a holomorphic (multi–valued) function. By [AGVII §13.1] the function $I^\omega_\zeta(t)$ can be expanded in a series

$$I^\omega_\zeta(t) = \sum_{\alpha,k} a_{\alpha,k} t^\alpha (\log t)^k \,,$$

where the numbers α are some greater then -1 rationals and k are natural numbers or 0. Fix $t_0 \in \dot{D}_\delta$. We identify $H^n(K_{t_0})$ with the continuous families of cycles. When we vary the family of cycles in $I^\omega_\zeta(t)$ we obtain so called *geometric section* of the cohomology Milnor fibre:

$$\mathrm{s}(\omega) = \sum_{\alpha,k} A_{\alpha,k} t^\alpha (\log t)^k,$$

with $A_{\alpha,k} \in H^n(K_{t_0})$. The smallest exponent α occurring in the expansion of $\mathrm{s}(\omega)$ is called the *order* of ω; it is denoted by $\alpha(\omega)$. The smallest possible order $\alpha_{\min}$ among all the forms ω is the order of $dz_0 \wedge \cdots \wedge dz_n$. The number $-(1+\alpha_{\min})$ is called the *complex oscillation index* of the singular point. The *principal part* of $\mathrm{s}(\omega)$ is the section

$$\mathrm{s}_{\max}(\omega) = \sum_k A_{\alpha(\omega),k} t^{\alpha(\omega)} (\log t)^k \,.$$

We have a control over all possible orders. For a holomorphic function f let $\boldsymbol{Supp}(f) \subset \mathbb{N}^{n+1}$ be the support of f, i.e. the set of the multiindices for which the corresponding monomial has nonzero coefficient in the Tylor expansion of f. Recall the definition of the Newton polyhedron [AGVII §6.2]:

$$\boldsymbol{\Gamma}(f) = conv \left(\bigcup_{(i_0,\dots i_n)\in \boldsymbol{Supp}(f)} \left((i_0,\dots,i_n) + \mathbb{R}^{n+1}_+\right) \right) \subset \mathbb{R}^{n+1} \,.$$

We introduce a valuation v which is associated to s:

$$v(f) = \sup \{ q \in \mathbb{R}_+ : \boldsymbol{\Gamma}(f) \subset q\, \boldsymbol{\Gamma}(s) \} \,. \tag{3.1}$$

For a differential form $\omega = g\, dz_0 \wedge \cdots \wedge dz_n$ we define [AGVII, §13.1]

$$v(\omega) = v(g\, z_0 \dots z_n) \,.$$

A consideration of an appropriate toric resolution of K leads to the following theorem:

Theorem 3.2. [AGVII, §13.1, Th. 2] *Suppose s has $\mathbb{C}$–nondegenerate principal part (in the sense* [AGVII, §6.2]*). Then:*
1. $\alpha(\omega) \geq v(\omega) - 1$. If $v(\omega) \leq 1$ then an equality holds,
2. the complex oscillation index is not greater then $-v(dz_0 \wedge \cdots \wedge dz_n)$. If $v(dz_0 \wedge \cdots \wedge dz_n) \leq 1$ then an equality holds.

The cohomology group $H^n(K_{t_0})$ is generated by the principal parts $\mathrm{s}_{\max}(\omega)$. Following [AGII, §13.2] we introduce a Hodge filtration

$$0 = F^{n+1} \subset F^n \subset \cdots \subset H^n(K_{t_0})$$

$$F^k = span\,\{[\mathrm{s}_{\max}(\omega)] \in H^n(K_{t_0}):\ \alpha(\omega) \le n-k\}\ .$$

We decompose the cohomology of K_{t_0}:

$$H^n(K_{t_0}) = \bigoplus_{\lambda-\text{eigenvalue}} H_\lambda\ .$$

into the eigenspaces of the action of the semisimple part of the monodromy. Denote by N the logarithm of the unipotent part of the monodromy. It defines a weight filtration in the following way: suppose $\sigma_1,\ \sigma_2,\ \dots\ \sigma_{\mu_\lambda}$ is a Jordan basis of H_λ with respect to N. Assume that the first block is of dimension d. Then $N(\sigma_i) = \sigma_{i-1}$ for $i \le d$. The weight filtration

$$0 = W_{-1,\lambda} \subset W_{0,\lambda} \subset \dots \subset W_{2n,\lambda} = H_\lambda$$

is defined block by block; e.g. the first one is

$$W_{n-1-d+2i,\lambda} \cap span\langle \sigma_1, \sigma_2, \dots \sigma_d\rangle = span\langle \sigma_1, \sigma_2, \dots \sigma_i\rangle \quad \text{for} \quad \lambda \ne 1\,,$$

$$W_{n-d+2i,\lambda} \cap span\langle \sigma_1, \sigma_2, \dots \sigma_d\rangle = span\langle \sigma_1, \sigma_2, \dots \sigma_i\rangle \quad \text{for} \quad \lambda = 1\,.$$

The resulting pair of filtrations on $H^n(K_{t_0})$ forms a mixed Hodge structure. The *spectrum* of the singular point is a set of rational numbers. They are of the form:

$$\big\{\alpha(\omega) \in \mathbb{Q}: \quad [\mathrm{s}_{\max}(\omega)] \in Gr^k F\, Gr_l W\,(H_\lambda)\,,\ [\mathrm{s}_{\max}(\omega)] \ne 0\,\big\}\ .$$

Each spectral number α coming from $Gr^k F\, Gr_l W\,(H_\lambda)$ satisfy: $\exp(2\pi i \alpha) = \lambda$ and $\alpha \in (n-k-1, n-k]$.

An another definition of the spectrum was given by Steenbrink via another Hodge filtration, see [St1], [St2].

The following is a consequence of the fact that W_* and F^* form a mixed Hodge structure on $H^n(K_{t_0})$.

Proposition 3.3. *Let $\sigma_1,\ \sigma_2,\ \dots\ \sigma_d \in H^n(K_{t_0})$ be a chain of elements satisfying $N(\sigma_i) = \sigma_{i-1}$ for $1 < i \le d$. Assume that $\sigma_d \in F^{m-d} \setminus F^{m-d+1}$ for some $m \in \mathbb{N}$. Then $\sigma_i \in F^{m-i} \setminus F^{m-i+1}$ for $i \le d$.*

Proof. The logarithm of the unipotent part of the monodromy

$$N : Gr_i W(H_\lambda) \to Gr_{i-2} W(H_\lambda)$$

shifts the Hodge filtration: $N(F^k Gr_i W(H_\lambda)) \subset F^{k-1} Gr_{i-2} W(H_\lambda)$, [AGV, 13.2 Lemma 12]. Proposition 3.3 follows from [AGVII, 13.2 Cor. 3] which says that N induces isomorphisms for $k,\ l \in \mathbb{Z}$:

$$N^l : F^k Gr_{n+l} W(H_\lambda) \xrightarrow{\simeq} F^{k-l} Gr_{n-l} W(H_\lambda)\,, \quad \text{if} \quad \lambda \ne 1\,,$$

$$N^l : F^k Gr_{n+1+l} W(H_\lambda) \xrightarrow{\simeq} F^{k-l} Gr_{n+1-l} W(H_\lambda)\,, \quad \text{if} \quad \lambda = 1\,. \quad \square$$

Since F^n is the smallest term of the Hodge filtration, then the forms of the order not greater then 0 contribute to the spectrum. Combining it with 3.2 we obtain:

Theorem 3.4. *Suppose s has $\mathbb{C}$–nondegenerate principal part. The set of spectral numbers which are nonpositive coincides with the numbers $v(\omega)-1$ for $v(\omega)\leq 1$. In particular these numbers can be read from the Newton polyhedron.*

Now we will prove a theorem which is the purpose of this paragraph.

Theorem 3.5. *If each residue class lifts to the intersection homology of K, then the number 0 does not belong to the spectra of the singular points of K.*

From the topological point of view this theorem is partially justified by the fact that the spectral numbers multiplied by $2\pi i$ are logarithms of the eigenvalues of the monodromy. Thus 0 must not be in the spectrum if 1 is not an eigenvalue of the monodromy. The second condition is equivalent for K to be a rational homology manifold; see Theorem 2.1.

Proof. Suppose that zero belongs to the spectrum of a point x. We will show, that there exists a form ω defined in a neighbourhood of x, such that $Res\,\omega$ restricted a link of x does not vanish in cohomology. By proposition 3.5 this implies that $[Res\,\omega]$ does not lift to intersection homology. We will find a cycle $\zeta\subset L=K\cap S_\epsilon(x)$ (where $S_\epsilon(x)$ is a small sphere around x), such that the integrale of the residue form on ζ does not vanish. From the assumption about the spectrum of x it follows that there exists nontrivial $\sigma\in F^n\cap H_{\lambda=1}$. Since F^n is the smallest term of the Hodge filtration, then by 3.3 the class σ is not contained in the image of N. We conclude that there exists an h_*–invariant cycle ζ_{t_0}, such that $\langle\sigma,\zeta_{t_0}\rangle=a\neq 0$. By 2.2 we know that ζ_{t_0} originates from a cycle $\zeta=\zeta_0\subset L$, i.e. one can include ζ in a continuous h_*–invariant family of cycles $\zeta_t\subset\partial K_t$ with $\zeta_0=\zeta$. By the choice made $\sigma=\mathrm{s}_{\max}(\omega)$ with $\alpha(\omega)=0$. Then the expansion of

$$I^\omega_\zeta(t)=\int_{\zeta_t}s\omega/ds$$

begins with $a_0=a$. Thus $\int_\zeta Res\,\omega=a\neq 0$. □

Remark. From the proof of 3.5 we see that (without any assumption on the spectrum) if $\alpha(\omega)>0$ then ω lifts to intersection homology.

The condition on spectrum in Theorem 3.5 can easily be read from the Newton polygon of s by Theorem 3.4. To check that 0 is not a spectral number for the most of the classified singularities see the table [AGVII, §13.3]. This shows that in general a residue class lies in intersection homology. For the singularities of the previous paragraph we have:

1) for $n\geq 2$ the spectrum does not contain 0 for the simple singularities;
2) for $n\geq 3$ the spectrum does not contain 0 for the unimodal parabolic singularities;
3) if $n=2$ then the spectrum contains 0 for the unimodal parabolic singularities.

All these functions are quasihomogeneous. In §§4–8 we treat this kind of singularities in details.

4. Valuation and quasihomogeneous functions

We will prove the converse of Theorem 3.5 for quasihomogeneous singularities.

Let $\mathbb{C}[z_0, \ldots, z_n]$ be the ring of the polynomials on $n+1$ variables. Let a_0, a_1, $\ldots$ a_n be a sequence of positive rational numbers called weights. Define a valuation (weight of a polynomial) $v : \mathbb{C}[z_0, \ldots, z_n] \to \mathbb{Q}$ by:
1) $v(z_i) = a_i$;
2) $v(f\,g) = v(f) + v(g)$;
3) if $f = \sum f_i$, f_i monomial, then $v(f) = \min\{\, v(f_i)\,\}$

If f is a sum of monomials of the same weight, then we say that f is quasihomogeneous (with respect to the valuation v). We extend the valuation for quasihomogeneous forms of the type:

$$\omega = \frac{f}{g} dz_{i_1} \wedge \cdots \wedge dz_{i_k}$$

setting $v(\omega) = v(f) - v(g) + a_{i_1} + \cdots + a_{i_k}$.

Suppose that at each singular point the hypersurface K is given by an equation $s = 0$ with s quasihomogeneous (in some coordinates and valuation). We rescale weights to obtain $v(s) = 1$. The obtained valuation is the one described in 3.1. For each singular point we define a number

$$\kappa = v(dz_0 \wedge \cdots \wedge dz_n) = \sum_{i=0}^{n} a_i \,.$$

By 3.2 the number $\kappa - 1$ is the complex oscillation index. We define a condition, which is equivalent to the condition on spectrum from Theorem 3.5.

Condition 4.1. *For any choice of nonnegative integers $k_i \in \mathbb{N} \cup \{0\}$, $i = 0, \ldots, n$ we have $\kappa + \sum k_i a_i \neq 1$.*

Of course the Condition 4.1 is satisfied if $\kappa > 1$.

Example 4.2. Let $s = z_0^3 + z_1^3 + z_2^4$. Then $a_0 = a_1 = \frac{1}{3}$ and $a_2 = \frac{1}{4}$. Then $\kappa = \frac{11}{12}$ but the Condition 4.1 is still satisfied.

5. A simple criterion of lift

We will prove the following:

Theorem 5.1. *Suppose that K of dimension n has isolated singularities given by quasihomogeneous equations in some coordinates. Let $\omega \in \mathcal{O}_M^{(n+1)}(K)$ be a meromorphic form with a first order pole on K. Suppose ω has no component of the weight 0 at each singular point. Then the residue class of ω lifts to intersection homology of K.*

The Theorem 5.1 is related to Theorem 3.5. We prove the Theorem 5.1 using few well known facts from the intersection homology theory. The reader is advised to compare the following proof with an example described in the Appendix.

Proof. By 1.2 one should show that $[Res\,\omega] \in ker\,(H^n(K^\circ) \longrightarrow H^n(\partial K^\circ))$, that is for each link L in K we have $[Res\,\omega_{|L}] = 0 \in H^n(L)$. The calculation is local, so from now on we assume that K is given by a quasihomogeneous equation $s = 0$.

The form ω can be written as $\omega = \frac{g}{s} dz_0 \wedge \cdots \wedge dz_n$. Now suppose that g is quasihomogeneous (otherwise we decompose g into a quasihomogeneous components).

By the assumption $v(g)+\kappa\neq 1$. We have a formula for $Res\,\omega$ at the points where $s_0=\frac{\partial s}{\partial z_0}\neq 0$ (see §1):

$$\omega=\frac{ds}{s}\wedge\frac{g}{s_0}dz_1\wedge\dots\wedge dz_n\,,$$

then

$$\boldsymbol{r}=\frac{g}{s_0}dz_1\wedge\dots\wedge dz_n$$

and $Res\,\omega=\boldsymbol{r}_{|K}$ at the points where $s_0\neq 0$. We have

$$v(\omega)=v(ds)-v(s)+v(\boldsymbol{r})=v(\boldsymbol{r})\,.$$

Then

$$\begin{aligned}v(\boldsymbol{r})=&v(g)-v(s)+v(dz_0)+\dots+v(dz_n)=\\=&v(g)-1+a_0+\dots+a_n=v(g)-1+\kappa\,.\end{aligned}$$

Let l be a natural number such that $l\,a_i\in\mathbb{N}$ for $i=0,\dots,n$. We construct a branched covering of $\mathbb{C}^{n+1}$:

$$\begin{aligned}\Phi:\mathbb{C}^{n+1}&\longrightarrow\mathbb{C}^{n+1}\\ \hat z_0,\dots,\hat z_n&\longmapsto\hat z_0^{la_0},\dots,\hat z_n^{la_n}\,.\end{aligned}$$

Let $\hat v$ be a standard valuation: $\hat v(f)=\deg f$ for homogeneous f. The map Φ has the property:

$$\hat v(\Phi^*\eta)=l\,v(\eta)$$

for any quasihomogeneous form η. We have

$$\hat v(\Phi^*\boldsymbol{r})=l(v(g)-1+\kappa)\,.$$

If we write $\Phi^*\boldsymbol{r}=q\,d\hat z_1\wedge\dots\wedge d\hat z_n$ then q is homogeneous function of degree

$$\hat v(q)=l(v(g)-1+\kappa)-n\,.$$

The mapping Φ is a branched covering of degree $l\,\kappa$. It induces a map of links:

$$\bar\Phi:\widehat{L}\to L\,,$$

where $\widehat{L}=\Phi^{-1}(K)\cap S_\epsilon$. Unfortunately $\widehat{L}$ may be singular; see Example 5.2. We have $H^*(K\setminus\{0\})\simeq H^*(L)$ and similarly we have $IH^{\underline{m}}_*(\Phi^{-1}(K)\setminus\{0\})\simeq IH^{\underline{m}}_{*-1}(\widehat{L})$ since $\Phi^{-1}(K)\setminus\{0\}=\widehat{L}\times\mathbb{R}_+$. To show that $[Res\,\omega]=0\in H^n(L)$ we will prove that $[\bar\Phi^*Res\,\omega_{|L}]=0\in IH^{\underline{m}}_{n-1}(\widehat{L})$. It is enough since the map

$$H^n(L)\xrightarrow{\bar\Phi^*}H^n(\widehat{L})\to IH^{\underline{m}}_{n-1}(\widehat{L})$$

is a monomorphism with a splitting

$$IH^{\underline{m}}_{n-1}(\widehat{L})\to H_{n-1}(\widehat{L})\xrightarrow{\bar\Phi_*}H_{n-1}(L)\xrightarrow{\simeq}H^n(L)\,.$$

The last map above is the inverse of the Poincaré duality isomorphism multiplied by $(l\,\kappa)^{-1}$. The maps to and from intersection homology are the canonical once.

To show vanishing in intersection homology we use a Gysin sequence of the fibration

$$S^1 \hookrightarrow \widehat{L} \xrightarrow{p} \widehat{L}/S^1$$

coming from the action of $\mathbb{C}^*$ on $\Phi^{-1}(K)$:

$$\to IH^{\underline{m}}_{n}(\widehat{L}/S^1) \xrightarrow{\cap e} IH^{\underline{m}}_{n-2}(\widehat{L}/S^1) \xrightarrow{p^*} IH^{\underline{m}}_{n-1}(\widehat{L}) \xrightarrow{p_*} IH^{\underline{m}}_{n-1}(\widehat{L}/S^1) \to .$$

The map $\cap e$ is the multiplication by the Euler class of the fibration; it is an isomorphism by hard Lefschetz since $\dim_{\mathbb{C}} \widehat{L}/S^1 = n-1$; [BBD]. We view $IH^{\underline{m}}_{n-1}(\widehat{L})$ as the L^2–cohomology of the nonsingular part of $\widehat{L}$:

$$IH^{\underline{m}}_{n-1}(\widehat{L}) = H^n_{(2)}(\widehat{L} \setminus \Sigma_{\widehat{L}}) =: H^n_{(2)}(\widehat{L})$$

for suitably chosen metrics on $\widehat{L} \setminus \Sigma_{\widehat{L}}$ and $(\widehat{L} \setminus \Sigma_{\widehat{L}})/S^1$; see [Ch], [We1]. Then the sequence has a form:

$$\to H^{n-2}_{(2)}(\widehat{L}/S^1) \xrightarrow{\cap e} H^n_{(2)}(\widehat{L}/S^1) \xrightarrow{p^*} H^n_{(2)}(\widehat{L}) \xrightarrow{p_*} H^{n-1}_{(2)}(\widehat{L}/S^1) \to .$$

The map p_* is just the integration along the fibers of p. Let us calculate the integral in the trivialization of the bundle $\mathbb{C}^{n+1} \setminus \{0\} \xrightarrow{p} \mathbb{P}^n$ over $U_0 = \{\hat{z}_0 \neq 0\} \subset \mathbb{P}^n$:

$$\begin{gathered} \mathbb{C}^* \times U_0 \longrightarrow p^{-1}(U_0)\,, \\ u_0, u_1, \ldots, u_n \longmapsto u_0, u_0u_1, \ldots, u_0u_n\,. \end{gathered}$$

We write $\Phi^* \boldsymbol{r}$ in u–coordinates:

$$\begin{aligned} \Phi^* \boldsymbol{r} &= q(\hat{z}_0, \ldots, \hat{z}_n) d\hat{z}_1 \wedge \cdots \wedge d\hat{z}_n = \\ &= u_0^{l(v(g)-1+\kappa)-n} \bar{q}(u_1, \ldots, u_n)(u_1 du_0 + u_0 du_1) \wedge \cdots \wedge (u_n du_0 + u_0 du_n) = \\ &= u_0^{l(v(g)-1+\kappa)-1} \bar{q}(u_1, \ldots, u_n) du_0 \sum_{i=1}^{n} (-1)^{i+1} u_i du_1 \wedge \cdots \overset{i}{\vee} \cdots \wedge du_n + \\ &\qquad\qquad + u_0 du_1 \wedge \cdots \wedge du_n = \\ &= u_0^{l(v(g)-1+\kappa)-1} du_0 \wedge \boldsymbol{r}_2 + \Theta\,, \end{aligned}$$

where $\boldsymbol{r}_2$ and Θ do not contain du_0 and $\boldsymbol{r}_2$ does not depend on u_0. Then

$$p_* \Phi^* \boldsymbol{r}(u_1, u_2) = \left(\int_{|u_0|^2(1+|u_1|^2+|u_2|^2)=1} u_0^{l(v(g)-1+\kappa)-1} du_0 \right) \boldsymbol{r}_2\,.$$

The integral can be nonzero only if $v(g) + \kappa = 1$. This is impossible by the assumption. Thus $p_* \Phi^*(Res\, \omega_{|L}) = 0$, so the residue lifts to intersection homology. $\square$

Example 5.2. Consider the polynomial

$$s(x, y, z) = (x + z^2)^2 + y^2 - z^4\,.$$

It has an isolated singularity of the type A_3. It is quasihomogeneous with weights $v(x) = v(y) = \frac{1}{2}$ and $v(z) = \frac{1}{4}$. The polynomial $\Phi^*(s)$ is:

$$\Phi^*(s) = (x^2 + z^2)^2 + y^4 - z^4 = x^4 + 2x^2 z^2 + y^4 \,.$$

Zero is not an isolated singularity since for $z = c = \text{const}$ we obtain:

$$x^4 + 2x^2c^2 + y^4 \sim x^2 + y^4$$

which is a singularity of the type A_3. An example of a singularity with $\widehat{L}$ nonsingular is $z_0^{k_0} + \cdots + z_n^{k_n}$ for any choice of $k_i \in \mathbb{N}$.

The Example 5.2 shows, that in the proof of the Theorem 4.2 we have to use the hard Lefschetz theorem for intersection homology instead of the standard one. We have used the hard Lefschetz of [BBD] for sake of brevity. Equally well we could work on the rational homology manifold L/S^1 contained in a weighted projective space.

6. Nonvanishing of the second residue

There is another way of looking at the calculation presented in the proof of the Theorem 5.1. Let the group $G = \mathbb{Z}/la_0 \times \cdots \times \mathbb{Z}/la_n$ acts on the coordinates of $\mathbb{C}^{n+1}$ by the multiplication by the roots of unity. Then $K = \widehat{K}/G$. We blow up $\widehat{K} \subset \mathbb{C}^{n+1}$ at 0 and obtain a diagram of varieties:

$$\begin{array}{ccccccccccc}
\widetilde{\mathbb{C}^{n+1}} & \supset & \widehat{Y} \cup \mathbb{P}^n & \stackrel{\widetilde{\Phi}}{\longrightarrow} & Y \cup \mathbb{P}(v) & = & \widehat{Y}/G \cup \mathbb{P}^n/G & \subset & \widetilde{\mathbb{C}^{n+1}}/G & & \\
{\scriptstyle \widehat{pr}}\big\downarrow & & \big\downarrow & & \big\downarrow & & & & {\scriptstyle pr}\big\downarrow & & \\
\mathbb{C}^{n+1} & \supset & \widehat{K} & \stackrel{\Phi}{\longrightarrow} & K & = & \widehat{K}/G & \subset & \mathbb{C}^{n+1}/G & = & \mathbb{C}^{n+1}.
\end{array}$$

Here $\mathbb{P}(v) = \mathbb{P}^n/G$ is the weighted projective space. We have $\widehat{Y} \cap \mathbb{P}^n = \widehat{L}/S^1$ and $Y \cap \mathbb{P}(v) = L/S^1$. The spaces $\mathbb{P}(v)$, Y and L/S^1 are rational homology manifolds, i.e. locally they are quotients of smooth manifolds by a finite group i.e. they are V–manifolds as defined by Steenbrink; [St1]. From the homology point of view they can be treated as ordinary (smooth) Kähler manifolds.

The last lines of the proof of the Theorem 5.1. lead to a definition of an element

$$res_2\omega = \left[\frac{1}{2\pi i}\int_p Res\, \omega_{|L}\right] \in IH^{\overline{m}}_{n-1}(\widehat{L}/S^1)\,.$$

This is an obstruction to lift the residue class to $IH^{\overline{m}}_n(K)$. We call it the *second residue*. The class $res_2\omega$ is G–invariant, so it is in

$$IH^{\overline{m}}_{n-1}(\widehat{L}/S^1)^G \simeq IH^{\overline{m}}_{n-1}(L/S^1) = H^{n-1}(L/S^1)\,.$$

We will show:

Theorem 6.1. *The second residue of* $\omega \in \mathcal{O}_M^{(n+1)}(K)$ *vanishes in* $H^{n-1}(L/S^1)$ *if and only if the component of* ω *of the weight* 0 *vanishes.*

Proof. The proof of 5.1 shows that the components of ω which have the weights different then 0 do not contribute to $res_2\omega$. Assume that $\omega = \frac{g}{s} dz_0 \wedge \cdots \wedge dz_n$ has the weight 0; i.e. $v(g) = 1 - \kappa$. We will show the nonvanishing of $res_2\omega$. The converse is obvious.

The class $res_2\omega$ is represented by the G invariant form $\boldsymbol{r}_{2|\widehat{L}/S^1}$. Since L/S^1 is V–manifold then its cohomology admits Hodge decomposition [St1, §1] and $res_2\omega$ is of the $(n-1,0)$–type. The form $\boldsymbol{r}_2$ is harmonic outside the singularities of L/S^1, therefore it vanishes in cohomology if and only if it is tautologically zero. We will show that the form $(\boldsymbol{r}_2)_{|\widehat{L}/S^1}$ does not vanish. We blow–up $\widehat{K} \subset \mathbb{C}^{n+1}$ at 0 (see the last diagram). We calculate the form $\Phi^*\omega$ pulled up to $\widetilde{\mathbb{C}^{n+1}}$ in the canonical coordinates (in the 0–th chart).

$$\widehat{pr}^*\Phi^*\omega = C\frac{\Phi^* g}{\Phi^* s}\left(\prod_{i=0}^{i=n} \hat{z}_i^{la_i-1}\right) d\hat{z}_0 \wedge \cdots \wedge d\hat{z}_n = \tag{6.2}$$

$$= C\frac{u_0^{l\,v(g)}\widetilde{\Phi^* g}}{u_0^l \widetilde{\Phi^* s}} u_0^{l\kappa-n-1}\left(\prod_{i=1}^{i=n} u_i^{la_i-1}\right) u_0^n \, du_0 \wedge \cdots \wedge du_n =$$

$$= C\frac{du_0}{u_0} \wedge \frac{\widetilde{\Phi^* g}}{\widetilde{\Phi^* s}}\left(\prod_{i=1}^{i=n} u_i^{la_i-1}\right) du_1 \wedge \cdots \wedge du_n \, ,$$

where $C = \prod_{i=0}^{i=n} la_i$. Here $\widetilde{p}(u_1, \ldots, u_n)$ denotes $p(1, u_1, \ldots, u_n)$. We see that the form $\widehat{pr}^*\Phi^*\omega$ has the first order pole on the exceptional divisor. The form $\boldsymbol{r}_{2|\widehat{Y}\cap\mathbb{P}^n}$ is the second Leray residue; [GS], [Le]. We can decompose the form $\widehat{pr}^*\Phi^*\omega$ in a way

$$\widehat{pr}^*\Phi^*\omega = \frac{du_0}{u_0} \wedge \frac{d\widetilde{\Phi^* s}}{\widetilde{\Phi^* s}} \wedge \boldsymbol{r}_2' \, , \tag{6.3}$$

where $\boldsymbol{r}_2'$ does not contain u_0 nor du_0. This is another expression of the second Leray residue of $\widehat{pr}^*\Phi^*\omega$. Thus $\boldsymbol{r}'_{2|\widehat{Y}\cap\mathbb{P}^n} = \boldsymbol{r}_{2|\widehat{Y}\cap\mathbb{P}^n}$. The function $\widetilde{\Phi^*(s)}$ describes $\widehat{Y} \cap \mathbb{P}^n$ in $\mathbb{P}^n$ for $u_0 \neq 0$, so to show that $r_{2|\widehat{Y}\cap\mathbb{P}^n} \not\equiv 0$ it suffices to check that $d\widetilde{\Phi^*(s)} \wedge r_2' \not\equiv 0$ on $\widehat{Y} \cap \mathbb{P}^n$. By the decompositions (6.2) and (6.3)

$$u_0\,\widetilde{\Phi^*(s)}\,\widehat{pr}^*\Phi^*\omega = du_0 \wedge d\widetilde{\Phi^* s} \wedge \boldsymbol{r}_2' = C\,du_0 \wedge \widetilde{\Phi^* g}\left(\prod_{i=1}^{i=n} u_i^{la_i-1}\right) du_1 \wedge \cdots \wedge du_n \, .$$

Since s does not divide g, thus $\widetilde{\Phi^* g}$ does not vanish on $\widehat{Y} \cap \mathbb{P}^n$. Moreover $\widehat{Y} \cap \mathbb{P}^n$ is not contained in any of the hyperplanes $u_i = 0$. Thus $d\widetilde{\Phi^* s} \wedge \boldsymbol{r}_2' \not\equiv 0$ on $\widehat{Y} \cap \mathbb{P}^n$ and hence $\boldsymbol{r}'_{2|\widehat{Y}\cap\mathbb{P}^n} \not\equiv 0$. □

As a corollary we obtain a result which implies Theorem 3.5 and its converse:

Corollary 6.4. *The Condition 4.1 is fulfilled at each singular point if and only if all the residue classes lift to intersection homology.*

Proof. The numbers $\kappa + \sum k_i a_i - 1$ are the possible weights of the forms from $\mathcal{O}_{\mathbb{C}^{n+1}}^{(n+1)}(K)$. By the Condition 4.1 it cannot be 0. Then by 6.1 the obstructions to lift vanish. □

Example 6.5. We compute the obstruction to lift for the Example 1.4 for which ($\kappa = 1$):

$$s(z_0, z_1, z_2) = z_0^3 + z_1^3 + z_2^3\,,$$
$$\omega = \frac{1}{s} dz_0 \wedge dz_1 \wedge dz_2\,.$$

Then $\widehat{L}/S^1 = L/S^1 \subset \mathbb{P}^2$. The second residue (i.e. the obstruction to lift) is:

$$res_2\omega = \left[\frac{1}{2\pi i}\int_p Res\,\omega\right] = \frac{1}{3}\left(u_1 du_2 - u_2 du_1\right)$$

in the notation used above. As one can check by hand, the integral

$$\int_{L/S^1 \cup \mathbb{RP}^2} res_2\omega \neq 0\,.$$

I the Appendix we will calculate $res_2\omega$ for an another form of the singularity P_8.

7. L^p–cohomology

To show that the residue form on the nonsingular part of K determines an element in intersection homology we apply the isomorphism of L^p-cohomology and intersection homology. It was proved by Cheeger for $p = 2$ and conjectured by [BGM] for arbitrary $p > 1$:

Theorem 7.1. [Ch], [We1]. *Let X be a pseudomanifold equipped with a Riemannian metric on the nonsingular part. Assume that this metric is concordant with a conelike structure of the pseudomanifold. If* $\operatorname{codim}\Sigma_K \geq 1 + \frac{1}{p-1}$*, then $H^*_{(p)}(X_{reg})$, the L^p–cohomology of the nonsingular part, is isomorphic to the intersection homology with respect to the perversity which is the largest perversity strictly smaller then the function $F(i) = \frac{i}{p}$.*

The perversity associated with $p \in [2, 2 + \frac{2}{n-1})$ is the middle perversity $\underline{m}$. Concordance with the conelike structure means that each singular point has a neighbourhood which is quasiisometric to the metric cone over the link, i.e. to $cL_x = L_x \times [0,1]/L_x \times \{0\}$ with the metric $t^2dx^2 + dt^2$. The intersection homology of a pseudomanifold K with isolated singularities is either $H^{2n-*}(K)$ or $H^{BM}_*(K)$ or the image of the Poincaré duality map $im(PD : H^{2n-*}(K) \xrightarrow{\cap[K]} H^{BM}_*(K))$. The case depends on the value of the perversity for $2n$. For the middle dimension we have:

Proposition 7.2. *If a hypersurface K with isolated singularities is equipped with a conelike metric and $n = \dim K > 1$ then*

$$H^n_{(p)}(K_{reg}) \simeq \begin{cases} H^{BM}_n(K) & \text{for} \quad 1 + \dfrac{1}{2n-1} \leq p < 2 \\ im\,PD & \text{for} \quad 2 \leq p < 2 + \dfrac{2}{n-1} \\ H^n(K) & \text{for} \quad 2 + \dfrac{2}{n-1} \leq p\,. \end{cases}$$

If the dimension is one then we should take the normalization of K instead of K.

In §8 we construct a suitable conelike metric and estimate the norm of a residue form for every $p > 1$. This way we will obtain a lifts of the residue classes to intersection homology for those manifolds which have singularities with $\kappa > 1$.

8. Local estimation

Suppose $K = \{s = 0\}$ is given globally in $\mathbb{C}^{n+1}$. We keep the notation of §4. We show the following:

Proposition 8.1. *Let s be a quasihomogeneous polynomial in $n+1$ variables with $v(s) = 1$. Assume that 0 is the only critical point of s. Then there exists a conelike metric on K_{reg} such that the norm of the residue form $Res\,\omega$ is L^p–integrable for all $\omega = \frac{g}{s}\,dz_0 \wedge \dots \wedge dz_n \in \mathcal{O}_M^{(n+1)}(K)$ with $v(\omega) > 0$.*

Proof. Assume that $s_0 = \frac{\partial s}{\partial z_0}$ does not vanish tautologically. Then at the points where $s_0 \neq 0$ we have

$$Res\left(\frac{g}{s}dz_0 \wedge \dots \wedge dz_n\right) = \boldsymbol{r}_{|K} = \left(\frac{g}{s_0}dz_1 \wedge \dots \wedge dz_n\right)_{|K}.$$

We choose $l \in \mathbb{R}_+$ and parameterize $\mathbb{C}^{n+1}$ by the homeomorphism:

$$(u_0, \dots, u_n) \longmapsto (u_0|u_0|^{l\,a_0-1}, \dots, u_n|u_n|^{l\,a_n-1}).$$

The set $\Phi^{-1}(K)$ is conical. We estimate the norm of the residue form in the metric induced by this parameterization. The norm $|\Phi^*(dz_i)|_u$ is (real) homogeneous of degree $l\,a_i - 1$, the denominator $\Phi^* s_0$ is homogeneous of degree $l(1-a_0)$. Thus the norm $|\Phi^*\boldsymbol{r}|_u$ is bounded by a homogeneous function of degree

$$l\,v(g) + \sum_{i=1}^{n}(l\,a_i - 1) - l(1-a_0) = l\,v(g) + \sum_{i=0}^{n} l\,a_i - n + l = l\,v(\omega) - n\,.$$

This estimation holds also at the points where $s_0 = 0$, for there is another derivative which does not vanish there. Then the integral $\int_{\{|u|=r\}\cap K} |\Phi^*\boldsymbol{r}|_u^p dz$ is bounded by a homogeneous function of degree

$$d = p\,(l\,v(\omega) - n) + 2n - 1 = p\,l\,v(\omega) + (2-p)n - 1$$

If $p = 2$ then we see that this function is integrable. For $p > 2$ in order to obtain $d > -1$ one should take $l > \frac{(p-2)n}{p\,v}$, where v is the smallest weight of ω which is greater then 0. □

Theorem 8.2. *If K has quasihomogeneous singularities with $\kappa > 1$ then the residue form defines an element in L^p–cohomology of K for a suitable metric.*

Proof. The minimal possible weight of a form $\omega \in \mathcal{O}_M^{(n+1)}(K)$ is $\kappa - 1$. At each singular point we choose l such that $l(\kappa - 1) > (p-2)n$. Then each residue form is L^p–integrable with respect to the conelike metric constructed in the proof of the Proposition 8.1. Hence it defines an element in L^p–cohomology. □

Observation 8.3. *The condition $\kappa > 1$ is fulfilled if the matrix of the second derivatives of s is of the rank at least 2 and $n > 1$.*

Proof. The polynomial s has either a term $z_i z_j$ or $z_i^2 + z_j^2$ so $a_i + a_j = 1$ and the remaining summands in κ are nonzero. □

Remarks. In the proof of Proposition 8.1 we can use the function $e^{-\frac{a_i}{|z_i|}}$ as well as $|z_i|^{l\,a_i}$ (l large). We obtain then a metric which is good for all $p>1$ at once. As a result we get the same condition for weights. Practically the theorem shows that we can integrate residue forms over chains which are regular enough i.e. which enter singular points along the cone lines. Thus the residue form defines a functional on the class of regular chains.

Below we list the singularities of §2 with computed weights and the numbers κ.

Type	weights	κ
A_k	$\frac{1}{k+1},\ \frac{1}{2},\ \frac{1}{2},\dots$	$\frac{n}{2}+\frac{1}{k+1}$
D_k	$\frac{k-2}{2k-2},\ \frac{1}{k-1},\ \frac{1}{2},\ \frac{1}{2},\dots$	$\frac{n}{2}+\frac{1}{2(k-1)}$
E_6	$\frac{1}{3},\ \frac{1}{4},\ \frac{1}{2},\ \frac{1}{2},\dots$	$\frac{n}{2}+\frac{1}{12}$
E_7	$\frac{1}{3},\ \frac{2}{9},\ \frac{1}{2},\ \frac{1}{2},\dots$	$\frac{n}{2}+\frac{1}{18}$
E_8	$\frac{1}{3},\ \frac{1}{5},\ \frac{1}{2},\ \frac{1}{2},\dots$	$\frac{n}{2}+\frac{1}{30}$
P_8	$\frac{1}{3},\ \frac{1}{3},\ \frac{1}{3},\ \frac{1}{2},\ \frac{1}{2},\dots$	$\frac{n}{2}$
X_9	$\frac{1}{4},\ \frac{1}{4},\ \frac{1}{2},\ \frac{1}{2},\dots$	$\frac{n}{2}$
J_{10}	$\frac{1}{3},\ \frac{1}{6},\ \frac{1}{2},\ \frac{1}{2},\dots$	$\frac{n}{2}$

We see that for all simple singularities we have $\kappa>1$ provided $n\geq 2$. For unimodal parabolic singularities one should take $n\geq 3$. If $n=2$ then by 6.1 the residue of $\frac{1}{s}dz_0\wedge\dots\wedge dz_n$ does not lift to intersection homology. The case P_8 is described in the Appendix (see also the Example 6.5).

9. Residues and $\mathcal{D}$-module $\mathcal{L}(K)$

The question of possibility of a lift of the residue class to intersection homology can be translated to the language of $\mathcal{D}$-modules. Fix some notation: there are the following sheaves on M:

$\mathcal{O}_M$ - the sheaf of holomorphic functions,
$\mathcal{O}_M(K)$ - the sheaf of meromorphic functions with the first order pole on K,
$\mathcal{O}_M(*K)$ - the sheaf of meromorphic functions with a pole on K of any order,
$\mathcal{O}_M^{(k)}$ - the sheaf of holomorphic forms of the type (k,0),
$\mathcal{O}_M^{(k)}(K)$ - the sheaf of meromorphic forms of the type (k,0) with the first order pole on K,
Ω_M^k - the sheaf of the complex valued C^∞ k-forms.

Consider the $\mathcal{D}$ module of meromorphic parts of functions with poles on K:

$$\mathcal{O}_M(*K)/\mathcal{O}_M=\mathcal{H}^1_K(\mathcal{O}_m)\,.$$

The de Rham functor acting from the category of regular holonomic $\mathcal{D}$-modules to perverse sheaves transforms $\mathcal{O}_M(*K)/\mathcal{O}_M$ to the complex of meromorphic forms modulo holomorphic forms (see e.g. [Bj, 1.2.17]):

$$DR(\mathcal{O}_M(*K)/\mathcal{O}_M)=(\mathcal{O}_M(*K)/\mathcal{O}_M)\otimes_{\mathcal{O}_M}\mathcal{O}_M^{(\bullet)}=\mathcal{O}_M^{(\bullet)}(*K)/\mathcal{O}_M^{(\bullet)}\,.$$

In the derived category it is isomorphic to the complex of semimeromorphic forms modulo C^∞-forms:

$$\left(\mathcal{O}_M(*K)\otimes_{\mathcal{O}_M}\Omega_M^\bullet\right)/\Omega_M^\bullet\,.$$

The module $\mathcal{O}_M(*K)/\mathcal{O}_M$ contains an unique simple submodule $\mathcal{L}(K)$ which is constant of rank 1 on K_{reg}, [Bj, 5.5.14]. De Rham functor transforms $\mathcal{L}(K)$ to intersection homology complex $IC^{\bullet}_{\underline{m}}(K)$ with a shift of degrees. (Intersection homology complex is considered as a sheaf on M supported by K.) We obtain:

Observation 9.1. *Suppose $\mathcal{O}_M(K)/\mathcal{O}_M \subset \mathcal{L}(K)$ then every closed form*

$$\omega \in \mathcal{O}_M(K) \otimes_{\mathcal{O}_M} \Omega^{k+1}_M$$

defines an element in

$$H^{k+1}(M; DR\mathcal{L}(K)) = IH_{2n-k}(K)\,.$$

For isolated singularities Vilonen [Vi] gives a description of $\mathcal{L}(K)$:

Theorem 9.2. *Let K be a hypersurface with an isolated singularity x. Then $h \in \mathcal{L}(K) \subset \mathcal{O}_M(*K)/\mathcal{O}_M$ if and only if*

$$\int_\xi h\omega = 0$$

for all $\xi \in H_{n+1}(B_\epsilon \setminus B_\epsilon \cap K)$ and all $\omega \in \mathcal{O}^{(n+1)}_M$, where B_ϵ is a small ball centered at x.

Let $\omega = \frac{g}{s} dz_0 \wedge \cdots \wedge dz_n \in \mathcal{O}^{(n+1)}_M(K)$. Then the class of ω belongs to the sheaf $\mathcal{L}(K) \otimes_{\mathcal{O}_M} \mathcal{O}^{(n+1)}_M \subset \mathcal{O}^{(n+1)}_M(*K)/\mathcal{O}^{(n+1)}_M$ if and only if

$$\int_\xi f\frac{g}{s} dz_0 \wedge \cdots \wedge dz_n = 0 \tag{9.3}$$

for all $\xi \in H_{n+1}(B_\epsilon \setminus B_\epsilon \cap K)$ and all $f \in \mathcal{O}_M$.

We have $H_{n+1}(B_\epsilon \setminus B_\epsilon \cap K) = H_n(S_\epsilon \cap K)$, where S_ϵ is a small sphere. The condition 9.3 can be reformulated:

$$\int_\zeta Res\left(f\frac{g}{s} dz_0 \wedge \cdots \wedge dz_n\right) = 0 \tag{9.4}$$

for all $\zeta \in H_n(S_\epsilon \cap K)$ and all $f \in \mathcal{O}_M$.

This condition is essentially stronger than the one in the Proposition 1.2, nethertheles we see that:

Theorem 9.5. *The conditions:*
1) $g\mathcal{O}_M(K)/\mathcal{O}_M \subset \mathcal{L}(K)$,
2) $\forall \omega \in g\mathcal{O}^{(n+1)}_M(K) \quad [Res\,\omega] \in im(PD : H^n(K) \to H^{BM}_n(K)) = IH^{\underline{m}}_n(K)$
are equivalent.

The Theorem 3.5 gives a necessary condition for 1) or 2) with $g = 1$. For quasihomogeneous singularities this is just the Condition 4.1 which is also a sufficient condition by 6.4.

10. Uniqueness of the lift

Denote by i the inclusion $M \setminus K \hookrightarrow M$. The sheaf of complex–valued C^∞ forms $\Omega^\bullet_{M\setminus K}$ is a soft resolution of the constant sheaf $\mathbb{C}_{M\setminus K}$. Thus $Ri_*\mathbb{C}_{M\setminus K} = i_*\Omega^\bullet_{M\setminus K}$. The inclusion i induces a distinguished triangle.

$$\begin{array}{ccccc} \mathbb{C}_M & \longrightarrow & Ri_*\mathbb{C}_{M\setminus K} & = & i_*\Omega^\bullet_{M\setminus K}\,. \\ {}_{+1}\nwarrow & & \swarrow & & \\ & R\Gamma_K\mathbb{C}_M & & & \end{array}$$

By taking the cohomology we get the long exact sequence of the pair $(M, M\setminus K)$. The stalk of $R\Gamma_K\mathbb{C}_M$ is:

$$\mathcal{H}^j_x(R\Gamma_K\mathbb{C}_M) \simeq H^j(B_x, B_x\setminus K) \xleftarrow[\simeq]{\cap[B_x]} H^{BM}_{2n+2-j}(K\cap B_x)\,,$$

where B_x is a small ball around x. Moreover, the whole sheaf $R\Gamma_K\mathbb{C}_M$ is isomorphic (with a shift of degrees) to the dualizing sheaf:

$$R\Gamma_K\mathbb{C}_M[2n+2] \simeq \mathbb{D}_K\,.$$

We obtain the residue morphism (Grothendieck residue)

$$res:\ i_*\Omega^\bullet_{M\setminus K}[2n+1] = Ri_*\mathbb{C}_{M\setminus K}[2n+1] \xrightarrow{+1} R\Gamma_K\mathbb{C}_M[2n+2] \simeq \mathbb{D}_K$$

which is an isomorphism on the cohomology sheaves for $j \neq -(2n+1)$

$$\begin{array}{ccccc} \mathcal{H}^j_x(i_*\Omega^\bullet_{M\setminus K}[2n+1]) & & \xrightarrow{\qquad res \qquad} & & \mathcal{H}^j_x(\mathbb{D}_K) \\ \| & & & & \| \\ H^{2n+1+j}(B_x\setminus K) & \xrightarrow{\delta} & H^{2n+2+j}(B_x, B_x\setminus K) & \xleftarrow[\simeq]{\cap[B_x]} & H^{BM}_{-j}(B_x\cap K)\,. \end{array}$$

In §8 we have constructed a conelike metric, for which (under the assumption $\kappa > 1$) Leray residue forms have p–integrable norms:

$$Res\left(\mathcal{O}^{(n+1)}_M(K)\right) \subset \mathcal{L}^n_{(p)}(K) \subset \Omega^n_{K_{reg}}\,.$$

The sheaf of L^p–cohomology $\mathcal{L}^*_{(p)}(K)$ is isomorphic to an appropriate intersection homology sheaf [We1]. This way we have constructed morphisms of sheaves:

$$\mathcal{O}^{(n+1)}_M(K)[n] \longrightarrow \mathcal{L}^\bullet_{(2)}(K)[2n] \simeq IC^\bullet_{\underline{m}}(K)$$

and

$$\mathcal{O}^{(n+1)}_M(K)[n] \longrightarrow \mathcal{L}^\bullet_{(p)}(K)[2n] \simeq IC^\bullet_{\underline{0}}(K)$$

for $p > 2 + \frac{2}{n-1}$. We use the convention that the sheaf $IC^\bullet_{\underline{m}}(K)$ is concentrated in negative degrees. We have described other conditions which allow to lift the sheaf morphism $\mathcal{O}^{(n+1)}_M(K)[n] \to \mathbb{D}_K$ to $\mathcal{O}^{(n+1)}_M(K)[n] \to IC^\bullet_{\underline{0}}(K)$. These are Condition 4.1 for quasihomogeneous singularities or an abstract condition 9.5 (for $g = 1$) for arbitrary singularities. We also have a necessary condition from 3.5. A question arises: is this lift unique? To be precise, let us consider the sequence of the canonical morphisms and the obstruction sheaves [GM, §4.5]:

$$\begin{array}{ccccccccc} \mathbb{C}_K[2n] & \simeq & IC^\bullet_{\underline{0}}(K) & \longrightarrow & IC^\bullet_{\underline{m}}(K) & \longrightarrow & IC^\bullet_{\underline{t}}(K) & \simeq & \mathbb{D}^\bullet_K\,. \\ & & {\scriptstyle +1}\nwarrow & \swarrow & {\scriptstyle +1}\nwarrow & \swarrow & & & \\ & & & S_1 & & S_2 & & & \end{array}$$

The triangles are distinguished in the derived category. The isomorphisms in the diagram follow from the fact that a hypersurface with isolated singularities is normal for $n>1$ ($H^0(L)=\mathbb{C}$). We regard these sheaves as sheaves on M supported by K. The cohomology of the links is nonzero only in dimensions 0, $n-1$, n and $2n-1$, so for arbitrary perversity the sheaf $IC^\bullet_{\underline{p}}(K)$ is isomorphic to:

1) $\mathbb{C}_K[2n]$ if $p(2n)<n-1$,
2) $IC^\bullet_{\underline{m}}(K)$ if $p(2n)=n-1$,
3) $\mathbb{D}_K$ if $p(2n)>n-1$.

The obstruction sheaves S_1 and S_2 are supported by the singular points and

$$\begin{aligned} \mathcal{H}^{-n-1}_x(S_1) = IH^{m}_{n+1}(cL_x) = H_n(L_x) \quad &\text{and} \quad \mathcal{H}^i_x(S_1)=0 \quad \text{for} \quad i\neq -(n+1), \\ \mathcal{H}^{-n}_x(S_2) = IH^{t}_{n}(cL_x) = H_{n-1}(L_x) \quad &\text{and} \quad \mathcal{H}^i_x(S_2)=0 \quad \text{for} \quad i\neq -n\,. \end{aligned}$$

Let $\mathcal{F}=\mathcal{O}^{(n+1)}_M(K)$. Applying the functor $RHom(\mathcal{F}[n],-)$ to the diagram above we obtain distinguished triangles and long exact sequences. Replacing R^0Hom by Hom_{D} — homomorphisms in the derived category we have:

$$\bigoplus_{x\in\Sigma_K} Hom(\mathcal{F}_x, H_n(L_x)) \longrightarrow Hom_{\mathrm{D}}(\mathcal{F}[n], IC^\bullet_{\underline{0}}(K)) \stackrel{\text{epi}}{\longrightarrow} Hom_{\mathrm{D}}(\mathcal{F}[n], IC^\bullet_{\underline{m}}(K))$$

$$Hom_{\mathrm{D}}(\mathcal{F}[n], IC^\bullet_{\underline{m}}(K)) \stackrel{\text{mono}}{\longrightarrow} Hom_{\mathrm{D}}(\mathcal{F}[n], IC^\bullet_{\underline{t}}(K)) \longrightarrow \bigoplus_{x\in\Sigma_K} Hom(\mathcal{F}_x, H_{n-1}(L_x))$$

This way we see that:

Proposition 10.1. *A lift of the residue morphism to $\mathcal{O}^{(n+1)}(K)[n]\longrightarrow IC^\bullet_{\underline{m}}(K)$ is unique in the derived category. If such a lift exists then there exists a lift to $\mathbb{C}_K[2n]$, which is not unique unless K is a rational homology manifold.*

The Proposition 10.1. is not a surprise since on the cohomology level we have

$$IH^{m}_{n}(K) = im\left(PD:\, H^n(K)\longrightarrow H^{BM}_n(K)\right)$$

for $n>1$. It seems that the lift to $IC^\bullet_{\underline{0}}(K)\simeq \mathcal{L}^\bullet_{(p)}(K)[2n]$ (p large) obtained in §8 essentially depends on the choice of a metric. We remind that a metric depends on the choice of coordinates in which the singularity is quasihomogeneous. The metric was determined by the weights.

Example 10.2. Consider the polynomial $s(x,y,z,t)=xy+y^{100}+z^2+t^2$. It is quasihomogeneous with weights $\frac{99}{100}$, $\frac{1}{100}$, $\frac{1}{2}$ and $\frac{1}{2}$. This is a Morse singularity (i.e. of type A_1), and one can change coordinates so that $s(x',y')={x'}^2+{y'}^2+z^2+t^2$. Then all weights are $\frac{1}{2}$.

11. Lift and mixed Hodge structure

Suppose that M is algebraic. The problem of lift is strictly connected with the mixed Hodge structure on $M \setminus K$. Consider the Deligne weight filtration on $H^k(M\setminus K)$, [De], [GS]:

$$0 = W_{k-1}H^k(M\setminus K) \subset W_kH^k(M\setminus K) \subset \cdots \subset W_{2k}H^k(M\setminus K) = H^k(M\setminus K)\,.$$

Suppose M is contained in a complete smooth manifold $\overline{M}$. A class $c \in H^k(M\setminus K)$ belongs to $W_kH^k(M\setminus K)$ if and only if it comes from $H^k(\overline{M})$. We will prove the following:

Proposition 11.1. *Suppose that a cohomology class c belongs to $W_{k+1}H^k(M\setminus K)$. Then the homological residue $res\, c \in H_{2n+1-k}(K)$ can be lifted to $IH^{\bar{m}}_{2n+1-k}(K)$.*

Remark. Of course this is not necessary condition for noncomplete M. The behavior of c at the infinity may cause that it does not belong to $W_{k+1}H^k(M\setminus K)$. Nevertheless for complete M the converse of 11.1 is not clear.

Proof. Consider a case when M is a complete manifold and $K = \bigcup_{i\in I} D_i$ is a sum of smooth divisors with normal crossings. Belonging to $W_{k+1}H^k(M\setminus K)$ means that c is represented by a form

$$\omega \in \Omega^{k-1}_M \wedge \Omega^1(\log\langle K\rangle)\,,$$

see e.g. [GS §5]. Then

$$Res\,\omega \in \bigoplus_{i\in I} \Omega^{k-1}_{D_i}$$

and

$$[Res\,\omega] \in \bigoplus_{i\in I} H^{k-1}(D_i) \simeq IH^{\bar{m}}_{2n+1-k}(K)\,.$$

The Proposition 11.1 remains true for an arbitrary singular variety. To prove this complete M, then take a resolution $\pi : \widetilde{M} \to \overline{M}$ of K and added boundary. We obtain $\widetilde{K} \subset \widetilde{M}$ with $\widetilde{M}\setminus\widetilde{K} \simeq M\setminus K$ and $\widetilde{K}$ is a sum of smooth components with normal crossings. If $c \in W_{k+1}H^k(M\setminus K)$ then $res\,c$ can be lifted to $IH^{\bar{p}}_{2n+1-k}(\widetilde{K})$. We can pull it down to $IH^{\bar{m}}_{2n+1-k}(K)$ by a map α which is constructed in [B^{2}FGK] (see also [We2]) in a way that the following diagram is commutative:

$$\begin{array}{ccccccc}
H^k(\widetilde{M}\setminus\widetilde{K}) & \stackrel{\delta}{\to} & H^{k+1}(\widetilde{M},\widetilde{M}\setminus K) & \simeq & H_{2n+1-k}(\widetilde{K}) & \leftarrow & IH^{\bar{m}}_{2n+1-k}(\widetilde{K}) \\
\| & & \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \alpha} \\
H^k(M\setminus K) & \stackrel{\delta}{\to} & H^{k+1}(M,M\setminus K) & \simeq & H_{2n+1-k}(K) & \leftarrow & IH^{\bar{m}}_{2n+1-k}(K).
\end{array}$$ □

Unfortunately this procedure uses desingularization, mixed Hodge structure and functoriality of intersection homology. Each of these ingredients is rather mysterious and hard to compute. We did not follow this direction. We plan to explain relation between these constructions in forthcoming papers. In this we have restricted our attention just to the case of isolated singularities.

12. Appendix: the P_8 singularity

Consider a singularity of type the P_8 in a form

$$s(z_0,z_1,z_2)=z_1^3+pz_0^2z_1+qz_0^3-z_0z_2^2$$

where p and q are real numbers such that z^3+pz+q does not have double roots. Let

$$\omega=\frac{1}{s}dz_0\wedge dz_1\wedge dz_2\,.$$

Then

$$s_2=\frac{\partial s}{\partial z_2}=-2z_0z_2$$

and

$$\boldsymbol{r}=-\frac{1}{2z_0z_2}dz_0\wedge dz_1$$

for $z_0z_2\neq 0$. We will apply the method of §§5–6 to compute the second residue. To calculate the cohomology of link $L=S^5\cap K$ we apply the Gysin exact sequence of the fibration

$$S^1\hookrightarrow L\stackrel{p}{\to}L/S^1\subset\mathbb{P}^2\,.$$

The projectivization L/S^1 of L is an elliptic curve in the projective plane $\mathbb{P}^2$, so it is a topological 2–dimensional torus. We obtain an exact sequence:

$$\to H^0(L/S^1)\stackrel{\cup e}{\longrightarrow}H^2(L/S^1)\stackrel{p^*}{\longrightarrow}H^2(L)\stackrel{p_*}{\longrightarrow}H^1(L/S^1)\stackrel{\cup e}{\longrightarrow}H^3(L/S^1)=0\,,$$

where the morphism p_* is the integration along the fibers of the projection p. The bundle $L\stackrel{p}{\to}L/S^1$ is the restriction of the tautological bundle $S^5\to\mathbb{P}^2$. Thus the Euler class of p is the restriction of the generator of $H^2(\mathbb{P}^2)$. Hence the evaluation of the Euler class $\langle e,[L/S^1]\rangle=deg\,s=3$. Thus rationally p_* in the Gysin sequence is an isomorphism. The necessary and sufficient condition to lift is vanishing of $\left[\int_p Res\,\omega_{|L}\right]\in H^1(L/S^1)$ as discussed in §6. Let

$$U_0=\left\{[z_0:z_1:z_2]\in\mathbb{P}^2\,:\,z_0\neq 0\right\}=\left\{[1:u_1:u_2]\in\mathbb{P}^2\,:\,u_1,\,u_2\in\mathbb{C}\right\}\simeq\mathbb{C}^2\,.$$

The tautological bundle $\tilde p:\mathbb{C}^3\setminus\{0\}\to\mathbb{P}^2$ restricted to U_0 is trivial:

$$\begin{aligned}\tilde p^{-1}(U_0)&\simeq\mathbb{C}^*\times\mathbb{C}^2\\(z_0,z_1,z_2)&\rightarrowtail\left(z_0,\left(\frac{z_1}{z_0},\frac{z_2}{z_0}\right)\right)\\(u_0,u_0u_1,u_0u_2)&\leftarrowtail(u_0,u_1,u_2)\end{aligned}$$

We write $\boldsymbol{r}$ in y–coordinates:

$$\boldsymbol{r}=-\frac{1}{2u_0^2u_2}du_0\wedge(u_0du_1+u_1du_0)=-\frac{1}{2u_0^2u_2}du_0\wedge u_0du_1=-\frac{du_0}{2u_0}\wedge\frac{du_1}{u_2}\,.$$

We integrate it over each fiber

$$p^{-1}([1,u_1,u_2])=\left\{(u_0,u_0u_1,u_0u_2):|u_0|^2(1+|u_1|^2+|u_2|^3)=1\right\}\;;$$

and obtain the second residue:

$$\boldsymbol{r}_2 = \frac{1}{2\pi i}\int_p \boldsymbol{r} = \frac{1}{2\pi i}\int_p \left[-\frac{du_0}{2u_0}\wedge\frac{du_1}{u_2}\right] = -\frac{1}{2}\frac{du_1}{u_2}\,.$$

We know that the form $\boldsymbol{r}_2$ does not vanish on L/S^1 by 6.1. Nevertheless we will integrate r_2 over a cycle Γ consisting of the real points of $L/S^1 \subset \mathbb{P}^3$ (or one of its components):

$$\int_\Gamma \boldsymbol{r}_2 = \pm 2\int_{u_{1\ \max}}^{\infty} \frac{du_1}{2\sqrt{u_1^3+pu_1+q}}\,,$$

where $u_{1\,max}$ is the biggest root of the polynomial $u_1^3+pu_1+q$. We have obtained an elliptic integral. The sign depends on the orientation of Γ.

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