Four paths from birational geometry to the elliptic genus

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Abstract. The article presents four reasons why the elliptic genus is the most general characteristic class that admits a generalization to singular spaces. We prove that the elliptic characteristic class (with an additional factor) is essentially the only characteristic class invariant under certain modifications, such as the Atiyah flop, Grassmannian flops, and modifications of Bott-Samelson resolutions. This result confirms and extends Totaro's result concerning the cobordism ring modulo classical flops. However, our approach is based on local calculus in equivariant cohomology.

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1. Introduction

Our goal is to address the following question:

Which characteristic classes can be reasonably extended to singular varieties?

Perhaps we wish to extend not to all types of singular varieties, but at least to a broad and meaningful family.

For smooth varieties, characteristic classes are typically defined via the tangent bundle. However, singular varieties lack a globally defined tangent bundle, which requires alternative constructions. Since the seventies of the last century, there have appeared various constructions of characteristic classes for complex singular varieties: first, the Chern-Schwartz-MacPherson classes, and later, the Todd-Baum-Fulton-MacPherson classes. Also there purely topological construction of L-classes followed from Goresky-MacPherson theory of intersection cohomology. In the 21st century motivic/Hirzebruch Chern classes

A.W. supported by Polish National Science Center grant number 2022/47/B/ST1/01896
MSC: 14C40, 14E30, 58J26, 11F50, 55N91.

were constructed by Brasselet-Schürmann-Yokura and finally the Borisov-Libgober elliptic classes. See the survey [23]. The most straightforward way to define a characteristic class for a singular variety is to find a resolution of singularities, compute its characteristic class, and apply the appropriate push-forward to the singular variety. In general, the result depends on the resolution chosen, and the contributions of singular loci must be accounted for as correction terms.

There are two common approaches to studying singularities. The first involves decomposing singular spaces into manifolds, called strata, and summing invariants that depend on each stratum and its associated normal data. In this work, we follow a different route: we replace the singular variety with a smooth one via resolution and then combine the invariants of the resolution with those of the components of the exceptional divisor. Any two resolutions of a singular variety $\psi_i : X_i \to Y$ are dominated by a third one

$$\begin{array}{ccccc} & {\scriptstyle\varphi_1} & X_3 & {\scriptstyle\varphi_2} & \\ & \swarrow & & \searrow & \\ X_1 & & & & X_2\,. \\ & \searrow & & \swarrow & \\ & {\scriptstyle\psi_1} & Y & {\scriptstyle\psi_2} & \end{array}$$

Therefore it is enough to understand invariants of smooth varieties and their behavior with respect to birational morphisms. By weak factorization theorem, [1], the morphisms φ_i can be assumed to be blow-ups in smooth centers.

By a *characteristic class*, we mean an element of the cohomology ring arising from the Hirzebruch formalism of multiplicative characteristic classes. To start we are interested in those characteristic classes $\mathfrak{c}$ that satisfy the following birational invariance property: for any birational proper morphism of complex algebraic varieties $\varphi\colon X' \to X$, we have

$$\varphi_* \mathfrak{c}(X') = \mathfrak{c}(X)\,. \tag{1}$$

Such classes are rare, as is well known and will also be demonstrated through a straightforward computation in §2. This scarcity suggests two possible approaches: either one must take into account the invariants of the exceptional locus, or restrict attention to a certain class of admissible resolutions. Both approaches will be discussed.

Our main result can be summarized as follows: we focus on the following settings

(1) In the first setting, instead of demanding the naive equality (1) we correct the formula by the contributions depending on the exceptional divisor. In fact, instead of characteristic classes of singular varieties we study relative characteristic classes of pairs consisting of a variety with a divisor. As in [6] we assume that the pair has log-canonical singularities (see Remark 3.2). We single out the condition imposed by the invariance with respect to the blow-up in a center of codimension two.
(2) In the second setting, we compare the characteristic classes of Bott-Samelson resolutions of a given Schubert variety. Here, we also consider the relative characteristic classes of pairs. We examine the consequences of the braid relation that connect two different resolutions.
(3) As it is common in birational geometry, we focus on a smaller class of singular varieties: those admiting a *crepant* resolution. We test what are the consequences of the Atiyah flop, i.e. the basic example of two, nonequivalent resolutions of the affine cone over $\mathbb{P}^1 \times \mathbb{P}^1$.
(4) Finally, we consider symplectic singularities. The best-known examples are the closures of nilpotent orbits in semisimple Lie algebras. Once again, we focus on the simplest possible cases of such singular varieties and examine the consequences of choosing two non-equivalent symplectic resolutions.

Theorem (Theorems 4.1, 7.1 10.1, 11.1)**.** *In each of the cases:*

(1) *pairs, with at worst log-canonical singularities*
(2) *Schubert varieties with Bott-Samelson resolutions,*
(3) *singularities admitting crepant resolutions*
(4) *symplectic singularities.*

the only (up to rescaling) characteristic class that is invariant under birational morphisms is the elliptic class multiplied by the factor $e^{\alpha c_1(X)}$ *for some* $\alpha \in \mathbb{C}$.

The true meaning of our theorem is that, by restricting the classes of singularities, we do not obtain any new invariant characteristic classes other than the elliptic class, up to a minor modification.

To place our result in a broader context, let us first recall the significant work of Totaro [25], who showed that the ideal in the cobordism ring $MU \otimes \mathbb{Q}$ consisting of almost complex manifolds with vanishing elliptic genus is generated by the difference $X_1 - X_2$, where X_1 and X_2 are related by a classical flop. This result is based on the previous study of the elliptic genus for singular varieties by Borisov and Libgober [6].

Theorem ([25, Theorem 4.1])**.** *Let* I *be the ideal in the complex bordism ring* $MU_* \otimes \mathbb{Q}$ *which is additively generated by differences* $X_1 - X_2$*, where* X_1 *and* X_2 *are smooth projective varieties related by a classical flop. Then the complex elliptic genus induces an isomorphism*

$$\mathbb{Q}[x_1, x_2, x_3, x_4] \to (MU_* \otimes \mathbb{Q})/I\,,$$

where $x_i = [\mathbb{P}^i]$.

It remains to note that the algebra in question is naturally isomorphic to the algebra of quasi-Jacobi forms of depth $(k, 0)$, $k \geq 0$, [20, Theorem 2.12]. Our result is similar in spirit, though it relies solely on a local approach and is formulated in terms of characteristic classes. The proof demonstrates that the constraints on admissible classes arise from comparing pairs of resolutions of particular singularities. That is consistent with the theorem quoted above. While the case of Atiyah flop can be deduced from Totaro's theorem, what appears to be a new result, is the case of symplectic singularities. We show that demanding invariance of a characteristic class under two independent Grassmannian flops determines the elliptic classes as well.

The idea of identifying conditions under which a general characteristic class remains invariant under specific modifications appears in [9], where the authors studied fundamental classes of Schubert varieties in generalized cohomology theories. There, it was shown that essentially only homological and K-theoretic fundamental classes are well-defined. Applying the generalized Riemann-Roch theorem [11, §42], the above fact translates into a statement about characteristic classes: only the Todd class (and its deformations) admits a generalization to Schubert varieties.

The scarcity of characteristic classes which are functorial on the nose for birational morphism can be overcome by modifying the notion of functoriality. By incorporating nontrivial contributions from the singular fibers into the definition of the pushforward, one can broaden the family of characteristic classes available for singular varieties. Most notably, the Chern-Schwartz-MacPherson classes should be mentioned. Furthermore, as shown in [8], Hirzebruch classes also admit generalizations to singular varieties. These can be interpreted as fundamental classes in hyperbolic K-theory [18]. A wide range of related characteristic classes for Schubert varieties (more precisely, Schubert cells) is presented in a series of papers by the authors of [2]. Various application to representation theory have been found.

Elliptic classes of varieties with at worst Kawamata log-terminal singularities were introduced by Borisov and Libgober in [6]. Moreover, defining characteristic classes for pairs turned out to be essential. As part of their construction, they defined a characteristic class for a smooth variety together with a $\mathbb{Q}$-divisor. This characteristic class was associated with a power series given by the Taylor expansion of the Jacobi theta function. Our guiding question regarding the most general form of characteristic classes for pairs was simply: Can the expanion of the theta function be replaced by another power series? The answer we obtained is that such a replacement is indeed possible: an additional factor can be introduced.

Our proof is based on elementary calculations once we adopt the perspective of equivariant cohomology. It is striking that simple and concrete computations can lead to such general result. The basic (and practically the only) tool is the Atiyah-Bott, Berline-Vergne localization theorem (see Theorem 2.1).

As and addendum to Theorem 4.1 we show that a function $f(x)$ defines a characteristic class of pairs, which is invariant with respect to blowups, if and only if $u(x) = \log\left(\frac{f(x)}{x}\right)''$ satisfies the differential equation

$$u''(x) = 12\left(\frac{u(x)}{x^2} + u(x)^2\right) + C\,. \tag{2}$$

Since the Borisov-Libgober elliptic genus is preserved by blow-ups, there is a solution of the form

$$u(x) = \sum_{k=1}^{\infty} \frac{\mathbb{G}_{2k+2}(\tau)}{(2k)!} x^{2k} \tag{3}$$

for $C = -5\mathbb{G}_4(\tau)$, see Corollary 6.1. Here $\mathbb{G}_{2k}(\tau)$ are modular forms, normalized as in [28, §2]. This allows us to write relations between the modular forms and express them easily as polynomials in $\mathbb{G}_4(\tau)$ and $\mathbb{G}_6(\tau)$. It was noticed by Anatoly Libgober, that the equation (2) is related to the basic equation for the Weierstrass $\wp$-function. Namely, (3) is essentially the regular part of the Weierstrass function

$$\wp(x) = x^{-2} + 2u(x)\,, \tag{4}$$

see see e.g. [24, p. 274]. Let us differentiate the equation for $\wp$

$$\wp'(x)^2 = 4\wp(x)^3 - g_2\wp(x) + g_3 \qquad |\tfrac{d}{dx}\,.$$

We obtain

$$2\wp''(x)\wp'(x) = \left(12\wp(x)^2 - g_2\right)\wp'(x)$$

and since $\wp'(x) \not\equiv 0$ we have

$$2\wp''(x) = 12\wp(x)^2 - g_2\,.$$

With the substitution (4), dividing by 4, we arrive to (2), the constant agrees: $5\mathbb{G}_4(\tau) = \frac{g_2}{4}$. Similar differential equation is obtained in §10, equation (11). Recursive relations for quasi-Jacobi forms follow. This is an alternative description of the 2-parameter elliptic genus, comparing with the characterization by Höhn [15, Lemma 2.2.1], [14, p. 198].

I am very grateful to Jakub Koncki for suggesting important improvements to the first version of this paper and for many inspiring discussions. Also, I would like to thank Anatoly Libgober for enlightening remarks and further discussion on elliptic genus.

2. Preliminaries and baby example

Let us recall Hirzebruch construction of characteristic classes. The *Hirzebruch formalism* provides a method to define *multiplicative characteristic classes* of complex vector bundles using formal power series. These classes appear naturally in various index theorems and in the definition of genera of manifolds. Let E be a complex vector bundle over a smooth manifold X. A *multiplicative characteristic class* $\mathfrak{c}$ assigns to E a cohomology class $\mathfrak{c}(E) \in H^*(M;\mathbb{C})$ satisfying:

$$\mathfrak{c}(E \oplus F) = \mathfrak{c}(E) \cdot \mathfrak{c}(F)$$

Such classes are defined in terms of the *Chern roots* $x_1, \ldots, x_n$ of E. Let $f(x) \in \mathbb{Q}[[x]]$ be a formal power series with constant term $f(0) = 0$, $f'(0) = 1$. Given a complex vector bundle E with Chern roots $x_1, \ldots, x_n$, the w define the multiplicative characteristic class associated to f by:

$$\mathfrak{c}_f(E) = \prod_{i=1}^{n} \frac{x_i}{f(x_i)}$$

This is a universal method to construct characteristic classes from formal power series. For a manifold X we define $\mathfrak{c}_f(X) = \mathfrak{c}_f(TX)$. The genus of a compact variety X associated to f is defined as the integral $\int_X \mathfrak{c}_f(X)$, or in the topological language – the push forward to the point.

Some classical classes and genera correspond to specific choices of $f(x)$:

series $f(x)$	characteritic class	genus	
$\frac{x}{1+x}$	Chern class	$\int_X c_*(TX)$	$\chi_{\rm top}(X)$
$1-e^{-x}$	Todd class	$\chi(X, \mathcal{O}_X)$	Todd genus
$\tanh x$	L-class	$\sigma(X)$	signature
$\frac{1-e^{-(1+y)x}}{1+ye^{-(1+y)x}}$	Hirzebruch class	$\chi_y(X)$	χ_y-genus

As a preview of the paper's main result, let us consider characteristic classes of smooth varieties that remain invariant under blow-ups. We make free use of equivariant cohomology, which makes even the simplest examples nontrivial – and, to our surprise, decisive. Let $bl : X' = \mathrm{Bl}_0(\mathbb{C}^2) \to \mathbb{C}^2 = X$ be the blow-up at the origin. The torus $\mathbb{T} = (\mathbb{C}^*)^2$ acts on $\mathbb{C}^2$ coordinatwise and on the resolution. Note that

$$H^*_{\mathbb{T}}(\mathbb{C}^2) = \mathbb{C}[x_1, x_2]\,.$$

Here x_1 and x_2 are the weights of the coordinates of $\mathbb{C}^2$. There are two fixed points in the blow-up. The corresponding weights of the torus action are the following

$$\{x_1,\ x_2 - x_1\} \qquad \text{and} \qquad \{x_2,\ x_1 - x_2\}\,.$$

Our basic tool to compute equivariant push-forward is the localization theorem:

Theorem 2.1 ([3]). *Let $\varphi : X \to Y$ be a $\mathbb{T}$-equivariant proper map of $\mathbb{T}$-manifolds and $\iota_X : X^{\mathbb{T}} \to X$, $\iota_Y : Y^{\mathbb{T}} \to Y$ be the inclusions of the fixed point sets. Denote by $e(\nu_{X^{\mathbb{T}}})$, $e(\nu_{Y^{\mathbb{T}}})$ the Euler classes of the normal bundles. Let S be the multiplicative system generated by characters of the normal bundles. The following diagram commutes:*

$$\begin{array}{ccc} H^*_{\mathbb{T}}(X) & \xrightarrow{e(\nu_{X^{\mathbb{T}}})^{-1}\iota_X^*} & S^{-1}H^*_{\mathbb{T}}(X^{\mathbb{T}}) \\ \varphi_* \downarrow & & \downarrow \varphi_*^{\mathbb{T}} \\ H^*_{\mathbb{T}}(Y) & \xrightarrow{e(\nu_{Y^{\mathbb{T}}})^{-1}\iota_Y^*} & S^{-1}H^*_{\mathbb{T}}(Y^{\mathbb{T}}) \end{array}$$

If there are finitely many fixed points then the map $\varphi_*^{\mathbb{T}}$ reduces to summation of the rational functions.

Torus localization for equvariant push forward leads to the formula:

$$\frac{bl_*\mathfrak{c}_f(X')}{x_1x_2} = \frac{1}{f(x_1)f(x_2-x_1)} + \frac{1}{f(x_1-x_2)f(x_2)} \in \hat{S}^{-1}H^*_{\mathbb{T}}(X) = \mathbb{Q}[[x_1,x_2]](x_1^{-1},x_2^{-1})\,,$$

while

$$\frac{\mathfrak{c}_f(X)}{x_1x_2} = \frac{1}{f(x_1)f(x_2)}\,.$$

The identity

$$\frac{1}{f(x_1)f(x_2-x_1)} + \frac{1}{f(x_2)f(x_1-x_2)} - \frac{1}{f(x_1)f(x_2)} = 0$$

after bringing to the common denominator we obtain

$$\mathcal{R} = f(x_2)f(x_2-x_1) + f(x_1)f(x_2-x_1) - f(x_1-x_2)f(x_2-x_1) = 0 \in \mathbb{Q}[[x_1,x_2]]$$

Let

$$\mathcal{DR} = \frac{d}{dx_2}\mathcal{R}|_{x_1=x,\ x_2=0}\,.$$

Assuming $f(0)=0,\ f'(0)=1$ we obtain

$$\mathcal{DR} = f(x) + f(-x)f'(x) = 0\,.$$

This relation allows to inductively find the coefficients of $f(x)$. We conclude that:

Corollary 2.2. *If the characteristic class satisfies $\varphi(\mathfrak{c}_f(X_1)) = \mathfrak{c}_f(X_2)$ for a blow up $\varphi\colon X_1 \to X_2$ in a codimension two center then*

$$f(x) = \frac{1-e^{-\nu x}}{\nu x} \quad \textit{for } \nu\in\mathbb{Q}\setminus\{0\} \quad \textit{or} \quad f(x)=x$$

i.e., $\mathfrak{c}_f$ is proportional to the Todd class or it degenerates to a trivial case.

Remark 2.3. Demanding the motivic scissor relation

$$bl_*\big(\mathfrak{c}_f(X') - \iota_*(\mathfrak{c}_f(E))\big) = \mathfrak{c}_f(X) - [0]\,,$$

see [8, Corollary 0.1], leads to a similar equation. We find that, up to a rescaling x, only the Hirzebruch class is motivic.

We have presented above calculation to illustrate our method in a more complex settings, where the computations were carried out symbolically, using Wolfram Mathematica. We provide the code in §12 so that readers can verify the calculations themselves. Further refinements include:

– incorporating divisor multiplicities and test for a single blow-up,

– restricting to special classes of singularities and test for the classical types of flops.

All the distinguished examples of resolutions lead to one conclusion. Only the elliptic characteristic class with an additional factor admit a generalization to singular varieties.

3. Relative characteristic classes

Let Y be a normal complex algebraic variety. It has a canonical divisor K_Y, which is a Weil divisor defined up to linear equivalence. Over the smooth locus $Y_{\rm reg}$, this divisor represents the line bundle of top-degree differential forms:

$$(K_Y)|_{Y_{\rm reg}} = \Omega^{\dim Y}_{Y_{\rm reg}}.$$

A variety Y is said to be $\mathbb{Q}$-Gorenstein if a multiple of the canonical divisor K_Y is Cartier. Then it defines an element of the Picard group $Pic(Y)\otimes\mathbb{Q}$, in particular it can be pulled back to a resolution of singularities. For a log-resolution $\psi : X \to Y$, the discrepancy divisor is defined as

$$E = K_X - \psi^*(K_Y).$$

If another resolution $\psi' : X' \to Y$ factors through a morphism $\varphi\colon X' \to X$, then:

$$\varphi^*(K_X - E) = K_{X'} - E', \text{ where } E' = K_{X'} - \psi'^*(K_Y).$$

The discrepancy divisor E can be uniquely written as a sum of irreducible components of the exceptional locus of φ

$$E = \sum_{i=1}^{r} a_i E_i,$$

where E_i are exceptional divisors and $a_i \in \mathbb{Q}$.

Many invariants of Y are defined using combinations of Chern classes of a resolution $c_*(TX)$, the fundamental classes $[E_i]$, and the coefficients a_i. Examples include:

- topological and Hodge-theoretic zeta functions [10],
- elliptic genus [6].

In a similar way the stringy Hodge numbers are defined, [4].

A study of a singular variety Y boils down to working with a system of smooth varieties X with SNC (simple normal crossing) divisors. By [1] we can reduce the argument to the case when the morphism between smooth varieties is a blow-up in a smooth center intersecting cleanly the divisor.

Let $\varphi\colon X' \to X$ be a birational morphism between smooth varieties. Given a divisor $E \subset X$, we define its *canonical pull-back* $\varphi^\kappa(E)$ by the condition:

$$K_{X'} - \varphi^\kappa(E) = \varphi^*(K_X - E)\,.$$

Example 3.1. Let $X = \mathbb{C}^2$ and $\varphi\colon X' \to X$ be the blow-up at 0. Let $E_i = \{z_i = 0\}$ for $i = 1, 2$ and let $E = \alpha_1 E_1 + \alpha_2 E_2$. Then the coefficient of the exceptional divisor in $\varphi^\kappa(E)$ is equal to $\alpha_1 + \alpha_2 + 1$.

Let $f(x) \in \mathbb{C}[[x]]$ be a formal power series with $f(0) = 0$, $f'(0) = 1$. Suppose (X, E) is a smooth variety with a simple normal crossing $\mathbb{Q}$-divisor $E = \sum_{i=1}^m a_i E_i$. We define a characteristic class of (X, E) depending on the Chern class $c_*(TX)$, the divisor classes $[E_i]$ and the coefficients a_i

$$\mathfrak{C}_f(X, E) \in H^*(X)\,.$$

The class $\mathfrak{C}_f(X, E)$ is defined as

$$\mathfrak{C}_f(X, E) = \prod_{i=1}^{\dim X} x_i F_f(x_i, h) \prod_{j=1}^{m} \frac{F_f(e_j, (a_j + 1)h)}{F_f(e_j, h)}\,,$$

where

$$F_f(x, h) = \frac{f(x + h)}{f(x) f(h)}\,.$$

Here x_i are the Chern roots and $e_i = [E_i]$.

Note that $TX = \bigoplus \mathcal{O}(E_i)$, then:

$$\mathfrak{C}_f(X, E) = \prod_{i=1}^{\dim X} e_i F_f(e_i, (a_i + 1)h)\,.$$

We say that $\mathfrak{C}_f$ is *preserved by canonical transformations* if for any birational proper morphism $\varphi\colon (X', E') \to (X, E)$ we have:

$$\varphi_*(\mathfrak{C}(X', \varphi^\kappa(E))) = \mathfrak{C}(X, E). \tag{5}$$

To test this property it is enough to test equality for blow-ups along smooth centers.

Remark 3.2. Note that in order to avoid zero in the denominator we have to assume that $a_i \neq -1$. Further taking blow-ups and applying canonical transformations the coefficients a_i can grow. Therefore to avoid zero we assume that $a_i > -1$. Then we say that the pair (X, E) has at worst log-canonical singularities.

4. Consequences of the blow-up

We search for a conditions under which the function f defines a class $\mathfrak{C}_f$ which is invariant under canonical transformations, i.e. it satisfies (5).

Theorem 4.1. *The following are equivalent*

(1) $\mathfrak{C}_f$ *is invariant with respect to canonical transformations.*
(2) *Blow-up relation holds:*

$$F_f(x,a+b)F_f(y-x,b)+F_f(y,a+b)F_f(x-y,a)=F_f(x,a)F_f(y,b)\,.$$

(3) *The function* $f(x)$ *is of the form* $e^{\lambda x+\mu x^2}f_0(x)$ *and* f_0 *satisfies the diffeterential equation*

$$3f_0''(x)^2-4f_0'(x)f_0^{(3)}(x)+f_0(x)f_0^{(4)}(x)+f_0(x)^2f_0^{(5)}(0)=0\,,$$
$$f_0(0)=f_0''(0)=f_0^{(3)}(0)=0\,,\ \ f'(0)=1\,.$$

Extending the coefficients to $\mathbb{C}$ *a generic*[1] *solution of the above equation is equal to:*

$$f(x)=e^{\lambda x+\mu x^2}\cdot\frac{\theta_\tau(\nu x)}{\nu\,\theta_\tau'(0)}$$

where $\theta_\tau(x)$ *is the Jacobi theta function,* $\tau\in\mathbb{H}^+$, $\lambda,\mu,\nu\in\mathbb{C}$.

Remark 4.2. First note that λ in the formulation of the theorem is completely irrelevant, since the factor $e^{\lambda x}$ cancels in the definition of F_f. We will show that $f(x)$ can be written as

$$f(x)=x\,e^{\lambda x+\mu x^2+\mathbf{r}(x)}$$

with $\mathbf{r}(x)=\sum_{k=2}^\infty a_{2k}x^{2k}$ an even function. Moreover the coefficients a_{2k} in the expansion of $\mathbf{r}(x)$ for $k>3$ depend on a_4 and a_6 polynomially, λ and μ are arbitrary. In addition to the expression given at the last line of the theorem one has to allow degenerate solutions, which we will discuss in §8.

Proof. Consider the blow-up of a codimension two center. Locally around the center the blow up looks like a family of blow-ups of surfaces at single points. Allowing families implies that the invariance with respect to the canonical transformation holds in equivariant cohomology. Then instead of an arbitrary center of codimension 2 it is enough to consider the blowup of $0\in\mathbb{C}^2$

$$\varphi:\ \ X'=\mathrm{Bl}_0\mathbb{C}^2\to\mathbb{C}^2=X\,.$$

Suppose x_1,x_2 are the weights of $\mathbb{T}=(\mathbb{C}^*)^2$ acting on $\mathbb{C}^2$. Let E_1 and E_2 be the coordinate divisors: the weight of the normal bundle to E_i is equal to x_i, as in Example 3.1. Define the divisor $E=\alpha_1E_1+\alpha_2E_2$. The canonical pull back of E has the multiplicity $\alpha_1+\alpha_2+1$ along the exceptional divisor. The invariance of $\mathfrak{C}_f$ with respect to the canonical transformation is equivalent to the identity

$$\begin{aligned}F_f(x_1,(\alpha_1+\alpha_2+2)h)F_f(x_2-x_1,(\alpha_2+1)h)+&\\ +F_f(x_2,(\alpha_1+\alpha_2+2)h)F_f(x_1-x_2,(\alpha_1+1)h)=&\\ =F_f(x_1,(\alpha_1+1)h)F_f(x_2,(\alpha_2+1)h)\,.&\end{aligned}$$

Setting the variables $x=x_1,\ y=x_2,\ a=(\alpha_1+1)h,\ b=(\alpha_2+1)h$ we obtain the identity stated in the condition (2) of Theorem 4.1. We note that the identity is satisfied for $F_f(x,h)=\frac{f(x+h)}{f(x)f(h)}$ if and only if it is satisfied for F_{f_0} with $f_0(x)=e^{-\lambda x-\mu x^2}f(x)$. Therefore we can substitute f by f_0 and assume that $f''(0)=f'''(0)=0$. Applying

[1] That is an open subset of solutions. Note that the dependence on τ is not algebraic.

the definition of F_f we obtain a rational expression. We bring the terms to a common denominator and then determine the numerator:

$$\begin{aligned}\mathcal{R}_0 = -f\,(a+b)\,f\,(x-y)\,f\,(y-x)\,f\,(a+x)\,f\,(b+y)\\ +\,f\,(b)\,f\,(x)\,f\,(y-x)\,f\,(a+x-y)\,f\,(a+b+y)\\ +\,f\,(a)\,f\,(x-y)\,f\,(y)\,f\,(a+b+x)\,f\,(b-x+y)\;.\end{aligned}$$

We compute

$$\frac{d^3}{dy\,da\,db}\mathcal{R}_0\Big|_{y=a=b=0}$$

Using the initial conditions:

$$f(0)=0,\quad f'(0)=1,\quad f''(0)=f'''(0)=0$$

we obtain:

$$f(x)\left(f(x)f'(-x)+f(-x)f'(x)\right)=f(x)f(-x)^2\left(\tfrac{f(x)}{f(-x)}\right)'=0\,.$$

Hence $f(x)/f(-x)$ is constant, and since $f(0)=0$, $f'(0)=1$ we deduce that $f(x)$ is an odd function. Let

$$\begin{aligned}\mathcal{R}=\mathcal{R}_0/f(x-y)=f\,(a+b)\,f\,(x-y)\,f\,(a+x)\,f\,(b+y)\\ -\,f\,(b)\,f\,(x)\,f\,(a+x-y)\,f\,(a+b+y)\\ +\,f\,(a)\,f\,(y)\,f\,(a+b+x)\,f\,(b-x+y)\;.\end{aligned}$$

This is a form of the Fay trisecant identity[2], [22, Example 2.10]. Using the antisymmetry

$$f(-x)=-f(x),\quad f'(-x)=f'(x),\quad f''(-x)=-f''(x),\quad f'''(-x)=f'''(x)$$

and vanishing of the first four derivatives at 0, we derive a differential equation: let us compute

$$\begin{aligned}\mathcal{DR}=\frac{d^6}{d^2y\,d^2a\,d^2b}\mathcal{R}\Big|_{y=a=b=0}=\\ =2\left(3f''(x)^2-4f'(x)f^{(3)}(x)+f(x)f^{(4)}(x)+f(x)^2f^{(5)}(0)\right).\end{aligned}\tag{6}$$

We write the solutions $f(x)$ in the form:

$$f(x)=xe^{\mathbf{r}(x)}\,,\ \mathbf{r}(x)=\sum_{i=2}^{\infty}a_{2i}x^{2i}\,.$$

Denoting d_j as the coefficient of x^j in the series expansion of $\mathcal{DR}$, we obtain:

$$\begin{aligned}d_4&=0\\ d_6&=288(6a_4^2+7a_8)\\ d_8&=720(12a_4a_6+11a_{10})\\ d_{10}&=144(24a_4^3+75a_6^2+140a_4a_8+143a_{12})\\ &\vdots\end{aligned}$$

The coefficient d_j is a polynomial in a_k. It is quasi-homogeneous of degree $j+2$, provided that we assign the weight i to a_i. We will show that the coefficient of a_{j+2} in d_j does not vanish for $j\geq 6$. Looking at the shape of the differential equation we see that the highest derivative is $f^{(4)}$. There are also terms $f'f^{(3)}$, $(f'')^2$ and a lower term f^2. Hence the

[2]The combinatorics of coefficients of the Fay trisecant identity was studied by Paweł Pielasa. He was able to show that the 7-th jet of f determines f.

coefficient of a_{j+2} in d_j is a polynomial of degree at most four. To find this polynomial we list few first values for $j \geq 6$:

$$2016, 7920, 20592, 43680, 81600, 139536, 223440, 340032, 496800, 702000, 964656, \dots .$$

We check that the sequence is described by the formula $2(j-4)(1+j)(2+j)(3+j)$, hence it does not vanish for $j > 4$. Therefore we can find the coefficients a_j recursively in terms of previous ones:

$$\begin{aligned}
a_8 &= -\tfrac{6}{7}a_4^2 \\
a_{10} &= -\tfrac{12}{11}a_4 a_6 \\
a_{12} &= \tfrac{3}{143}(32a_4^3 - 25a_6^2) \\
a_{14} &= \tfrac{1440}{1001}a_4^2 a_6 \\
a_{16} &= -\tfrac{36}{2431}(44a_4^4 - 75a_4 a_6^2) \\
a_{18} &= -\tfrac{60}{46189}(1392a_4^3 a_6 - 275a_6^3) \\
a_{20} &= \tfrac{432}{2540395}(3872a_4^5 - 12125a_4^2 a_6^2)
\end{aligned}$$

There are no conditions for a_4 and a_6.

To sum up: The structure of the differential equation and the expansion of the function $f(x)$ in exponential form leads to a recursive algebraic formula for the coefficients. There are two degrees of freedom. Additionally we can multiply $f(x)$ by $e^{\mu x^2}$ with arbitrary coefficient μ. The irrelevant factor $e^{\lambda x}$ does not count. Now the point is that we know at least one parameter family of solutions. Let $\theta_\tau(x)$ be the Jacobi theta function. By [7, Theorem 3.6] the characteristic class $\mathfrak{C}_f$ for $f(x) = \frac{\theta_\tau(x)}{\theta'_\tau(0)}$ is preserved by the canonical transformation. After a brief recollection of the basic facts about modular forms in the next section we will produce a three parameter family of solutions. That will conclude the proof of the Theorem 4.1.

5. Jacobi theta function and modular forms

Let us recall the basic facts about Jacobi theta function. We adopt the convention of [28]. Let us define the Jacobi theta function by the triple product formula

$$\theta_\tau(x) = q^{1/8}(e^{x/2} - e^{-x/2}) \prod_{n=1}^{\infty}(1-q^n)(1-q^n e^x)(1-q^n e^{-x}) .$$

As usual $q = e^{2\pi i \tau}$, $\mathrm{im}(\tau) > 0$. The theta function can be written as

$$\theta_\tau(x) = \theta'_\tau(0)\, x \,\mathrm{Exp}\Big(-2\sum_{k=1}^{\infty} \frac{\mathbb{G}_k(\tau)}{k!} x^k\Big) , \tag{7}$$

where $\mathbb{G}_k(\tau)$ are normalized modular forms

$$\mathbb{G}_k(\tau) = \begin{cases} -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sum_{d|n} d^{k-1} q^n & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

see [28, p.450]. Here B_k is the Bernoulli number.

The function $f(x) = \frac{\theta_\tau(x)}{\theta'_\tau(0)}$ satisfies the condition (1) of Theorem 4.1. The coefficients a_{2k} appearing in the proof of Theorem 4.1 are of the form

$$a_{2k} = -2\frac{\mathbb{G}_{2k}(\tau)}{(2k)!} .$$

Let us quote the basic fact about modular forms:

Theorem (see [29, §2, Prop. 4])**.** *The modular forms $\mathbb{G}_4(\tau)$ and $\mathbb{G}_6(\tau)$ are algebraically independent. The $\mathbb{Q}$-algebra generated by $\mathbb{G}_4(\tau)$ and $\mathbb{G}_6(\tau)$ contains all $\mathbb{G}_k(\tau)$ for $k \geq 8$.*

Together with $\mathbb{G}_2(\tau)$ the forms $\mathbb{G}_4(\tau)$ and $\mathbb{G}_6(\tau)$ can be treated as free variables. If we replace $a_2 = -\mathbb{G}_2(\tau)$ in $f(x)$ by $-\mathbb{G}_2(\tau) + \mu$ the invariance of $\mathfrak{C}_f$ for any blow-up remains preserved since it is an algebraic relation involving coefficients a_{2k}, $k \geq 1$. Together with possibility of rescaling x we have three degrees of freedom. The generic solution is of the form

$$f(x) = e^{\mu x^2} \frac{\theta_\tau(\nu x)}{\nu \theta'_\tau(0)}$$

with $\mu, \nu, \tau \in \mathbb{C}$, $\nu \neq 0$, $im(\tau) > 0$. Multiplying by the factor $e^{\lambda x}$ does not change the class $\mathfrak{C}_f$. This concludes the proof of Theorem 4.1. □

The modular form $\mathbb{G}_2$, having anomalous transformation behavior, often introduces complications in modular-invariant constructions. However, it does not appear in the elliptic genera of Calabi–Yau manifolds, which ensures that such genera remain weak Jacobi modular forms. As we have shown, $\mathbb{G}_2$ does not play a role in the constraints that define admissible functions $f(x)$, further indicating its irrelevance in this context. If necessary, $\mathbb{G}_2$ can be artificially eliminated, by multiplication by the factor $e^{\mathbb{G}_2(\tau)x^2}$.

6. Differential equation

Our proof of Theorem 4.1 is based on the analysis of the differential expression (6) vanishing after the substitution $f(x) = e^{\mathbb{G}_2 x^2} \frac{\theta_\tau(x)}{\theta'_\tau(0)}$. Remarkably simpler differential equation is satisfied by the function $v(x)$ after the substitution $f(x) = x\, e^{-2v(x)}$:

$$12\left(x^2 v''(x)^2 + v''(x)\right) - x^2\left(v^{(4)}(x) + 5v^{(4)}(0)\right) = 0\,. \tag{8}$$

The function

$$v(x) = \sum_{k=2}^{\infty} \frac{\mathbb{G}_{2k}(\tau)}{(2k)!} x^{2k}\,.$$

is a solution. We note that $v(x)$ and $v'(x)$ do not appear in the equation (8). Let us set $u(x) = v''(x)$ and assume $u(0) = 0$. Finally we obtain quite elegant form of a differential equation characterizing the sequence of the modular forms:

Corollary 6.1. *The function*

$$u(x) = \sum_{k=1}^{\infty} \frac{\mathbb{G}_{2k+2}(\tau)}{(2k)!} x^{2k}\,.$$

satisfies the equation

$$u''(x) = 12\left(\frac{u(x)}{x^2} + u(x)^2\right) + C$$

with $C = -5\mathbb{G}_4(\tau)$.

As it was explained in the end of Introduction this equation can be derived directly from the basic differential equation involving the Weierstrass $\wp$-function.

Remark 6.2. From the above equation we obtain a convenient induction allowing to compute $\mathbb{G}_{2k}(\tau)$ in terms of $\mathbb{G}_{2i}(\tau)$ for $i < k$. A substantially different set of relations involving $\mathbb{G}_{2k}(\tau)$ (with a different normalization) has was proven in [21].

7. Bott-Samelson resolutions of Schubert varieties

An important class of singular varieties is formed by Schubert varieties, subvarieties of rational homogeneous spaces G/P. In the case of complete flag varieties, Schubert varieties X_w are indexed by elements of the Weyl group. Each has a distinguished set of resolutions, known as a Bott–Samelson resolutions, which is indexed by the reduced expression of the corresponding Weyl group element in terms of simple reflections. Any two reduced expressions of the same element differ by a sequence of braid relations. Therefore, verifying the independence of the Bott–Samelson resolution reduces to checking invariance under the braid relations. This condition was discussed in [22, Equation (32-33)]. The formula to be checked is:

$$\begin{aligned}F_f(z_2-z_1,\mu_3-\mu_2)F_f(z_3-z_2,\mu_3-\mu_1)F_f(z_2-z_1,\mu_2-\mu_1)+\\ +F_f(z_1-z_2,h)F_f(z_3-z_1,\mu_3-\mu_1)F_f(z_2-z_1,h)=\\ =F_f(z_3-z_2,\mu_2-\mu_1)F_f(z_2-z_1,\mu_3-\mu_1)F_f(z_3-z_2,\mu_3-\mu_2)+\\ +F_f(z_2-z_3,h)F_f(z_3-z_1,\mu_3-\mu_1)F_f(z_3-z_2,h)\,. \quad (9)\end{aligned}$$

Here $z_1, z_2, z_3, \mu_1, \mu_2, \mu_3$ and h are free variables. Without a loss of generality we can assume that $z_1 = \mu_1 = 0$, since in the formula only the differences $z_i - z_j$ and $\mu_i - \mu_j$ appear. The above formula is a part of the braid relation satisfied by the elliptic Demazure-Lusztig operators, [26, Theorem 11.1]. For twisted motivic Chern classes it was discussed in [17, Proposition 11.9]. This relation already appears when checking that two resolutions of the pair $(X_w, \partial X_w)$ give equal equivariant characteristic classes for $w = s_1s_2s_1 = s_2s_1s_2$ in the Weyl group of $SL_3(\mathbb{C})$.

It is surprising that we do not obtain more solutions for f than in the blow-up case. The computations are similarly elementary; however, due to the length of the intermediate steps, we omit them here. Instead, we provide the Wolfram Mathematica code (see §12). Interestingly, the resulting differential equation is identical to that in the blow-up case, up to multiplication by an invertible factor.

Theorem 7.1. *The function f defines a characteristic class $\mathfrak{C}_f$ which does not depend on the choice of Bott-Samelson resolution of Schubert varieties if and only if f satisfies the equivalent conditions of Theorem 4.1.*

This means that no additional characteristic classes can be defined for Schubert varieties beyond those permitted for general Kawamata log-terminal pairs.

8. Degenerations

Suppose $f(x) = \frac{\theta_\tau(x)}{\theta'_\tau(0)}$. If $\tau \to i\infty$, i.e., $q \to 0$. By [28, §3] the Fourier expansion of F_f is following:

$$F_f(x,h) = \frac{1-(ab)^{-1}}{(1-a^{-1})(1-b^{-1})} + \sum_{n=1}^{\infty}\left(\sum_{d|n}(a^{-d}b^{-n/d} - a^d b^{n/d})\right) q^n$$

where $a = e^x$, $b = e^h$. As $q \to 0$, the modular form $\mathbb{G}_k$ converges to $\frac{B_k}{2k}$. The limit of F_f we see immediately from the expansion above

$$F_f(x,h) \to \frac{(1-e^{-(x+h)})}{(1-e^{-x})(1-e^{-h})}$$

This degeneration is related to the Hodge-theoretic zeta function. After a suitable change of variables, the integral $\int_X \mathfrak{C}_f(X, sD)$ corresponds to the Hodge-theoretic specialization of the global motivic zeta function associated with the divisor D. The motivic global zeta function has been defined, in considerable generality, for $\mathbb{Q}$-divisors on $\mathbb{Q}$-Gorenstein varieties in [19].

A particularly interesting degeneration arises when we set $q = e^{-h} \to 0$; see [16], where the corresponding constructions are described in the context of K-theory. Applying the Riemann–Roch transformation allows us to transfer these classes to cohomology. In this limit, the resulting classes depend discontinuously on the coefficients of the divisor, the dependence is locally constant.

Another interesting degeneration cannot be obtained solely from the elliptic characteristic class. By treating a_2, a_4, and a_6 as free variables, we are free to set $a_2 = \mu$ and $a_4 = a_6 = 0$. Then

$$f(x) = xe^{\mu x^2}, \quad xF_f(x,h) = e^{2\mu h x}(1 + \tfrac{x}{h}) .$$

Assuming that the boundary divisor is empty and X is smooth the associated characteristic classes we can be writen as

$$\mathfrak{C}_f(X, \emptyset) = e^{2\mu c_1(X)}\, c_*(X)$$

assuming that $h = 1$. It is remarkable that this class can be extended to singular varieties.

9. A nice choice of generators of quasi-Jacobi forms

In this section instead of cohomology with rational coefficients we consider complex coefficients. From the work of Borisov and Libgober it follows that the elliptic classes, defined in terms of the Chern roots

$$\mathcal{E}\ell\ell(X) = \prod_{i=1}^{\dim X} x_i \frac{\theta_\tau(x_i + h)\theta'_\tau(0)}{\theta_\tau(x_i)\theta_\tau(h)}$$

has a generalization to singular varieties. In particular if $\psi_i \colon X_i \to Y$, $i = 1, 2$ are two crepant resolutions of a singular variety, then $\psi_{1*}\mathcal{E}\ell\ell(X) = \psi_{2*}\mathcal{E}\ell\ell(X)$. The elliptic genus of a variety admitting a crepant resolution belongs to the algebra spanned by the coefficients of the expansion

$$x\, F_\tau(x,h) = x\frac{\theta_\tau(x + h)\theta'_\tau(0)}{\theta_\tau(x)\theta_\tau(h)} = 1 + \sum_{i=1}^{\infty} \phi_i(\tau,h)x^i \,.$$

This algebra is a polynomial algebra freely generated by $\phi_1, \phi_2, \phi_3, \phi_4$, equal to the algebra of quasi-Jacobi forms (of depth $(k, 0)$, $k > 0$), see [20, Theorem 2.12]. In our approach it is more natural to use another set of generators:

$$x\, F_\tau(x,h) = \frac{x}{f(x)} = \operatorname{Exp}\Big(-\sum_{i=1}^{\infty} \widetilde{E}_i(\tau,h)\Big)\,,$$

$$\begin{aligned}
\widetilde{E}_1(\tau,h) &= -\phi_1,\\
\widetilde{E}_2(\tau,h) &= \tfrac{1}{2}\left(\phi_1^2 - 2\phi_2\right),\\
\widetilde{E}_3(\tau,h) &= \tfrac{1}{3}\left(-\phi_1^3 + 3\phi_2\phi_1 - 3\phi_3\right),\\
\widetilde{E}_4(\tau,h) &= \tfrac{1}{4}\left(\phi_1^4 - 4\phi_2\phi_1^2 + 4\phi_3\phi_1 + 2\phi_2^2 - 4\phi_4\right).
\end{aligned}$$

The functions $\widetilde{E}_k(\tau,h)$ can be computed from the formula (7) or directly from [28, p. 456, (viii)]

$$F_\tau(x,h) = \frac{x+h}{xh}\operatorname{Exp}\left(-\sum_{k>0}\frac{2}{k!}\big((x+h)^k - x^k - h^k\big)\mathbb{G}_k(\tau)\right)$$

(we recall that $\mathbb{G}_k(\tau)=0$ for k odd). Thus

$$\sum_{i=1}\widetilde{E}_i(\tau,h)x^n=\log\Big(\frac{h}{h+x}\Big)+\sum_{k>0}\sum_{i=1}^{k-1}\frac{2}{k!}\binom{k}{i}x^ih^{k-i}\mathbb{G}_k(\tau)\,,$$

$$\widetilde{E}_i(\tau,h)=\frac{(-1)^i}{i}\frac{1}{h^i}+\sum_{k>i}\frac{2}{k!}\binom{k}{i}\mathbb{G}_k(\tau)h^{k-i}\,. \tag{10}$$

Note that $\mathbb{G}_2(\tau)$ appears only in $\widetilde{E}_1(\tau,h)$. As in [27, Ch. III, eq. (10-11)], let us set

$$e_k(\tau)=\frac{2}{(k-1)!}\mathbb{G}_k(\tau)\,,$$

$$E_i(\tau,h)=\frac{1}{h^i}+(-1)^i\sum_{k=1}^{\infty}\binom{k-1}{i-1}e_k(\tau)h^{k-i}\,.$$

In fact, the summation is over $k\geq i$. The equation (10) can be transformed as follows

$$\widetilde{E}_i(\tau,h)=\frac{(-1)^i}{i}\left(\frac{1}{h^i}+(-1)^i\sum_{k>i}\frac{2}{(k-1)!}\binom{k-1}{i-1}\mathbb{G}_k(\tau)h^{k-i}\right)\,,$$

$$\widetilde{E}_i(\tau,h)=\frac{(-1)^i}{i}\Big(E_i(\tau,h)-e_i(\tau)\Big)\,.$$

If the following section we consider a general formal power series

$$f(x)=x\,\mathrm{Exp}\Big(\sum_{i=1}^{\infty}b_ix^i\Big),$$

and investigate for which sequences $\{b_i\}_{i=1}^{\infty}$ the associated characteristic class $\mathfrak{c}_f$ satisfies $\psi_{1*}\mathfrak{c}_f(X)=\psi_{2*}\mathfrak{c}_f(X)$, for two crepant resolutions of a singular variety. We treat the coefficients b_i purely formally, without reference to their origin in Jacobi forms. Ultimately, we conclude that the relations among admissible sequences $\{b_i\}$ coincide with those governing quasi-Jacobi forms.

As before, computations are performed in equivariant cohomology. As our initial example of a singular space, we consider the affine cone over $\mathbb{P}^1\times\mathbb{P}^1$; later, we turn to nilpotent cones.

10. Atiyah Flop

We consider two small (or in general crepant) resolutions of a given singular variety and require that the push forward of the characteristic classes $\mathfrak{c}_f(X_1)$ and $\mathfrak{c}_f(X_2)$ coincide. As a test case, we examine the simplest possible situation – the Atiyah flop. The following calculation was suggested in [13]. Let V and W be vector spaces of dimension 2 and let

$$Y=\{A\in\mathrm{Hom}(V,W)\;:\;rk(A)\leq 1\}\subset\mathrm{Hom}(V,W)\simeq\mathbb{C}^4\,.$$

There are two natural small resolutions of Y, determined by the general structure of the fixed-rank locus:

$$\begin{array}{ccccc}
X_1 & & Y & & X_2\\
\mathrm{Hom}(\mathcal{O}(1),W) & \xrightarrow{\ \psi_1\ } & \left\{\begin{array}{c}A:V\to W\\ rk(A)\leq 1\end{array}\right\} & \xleftarrow{\ \psi_2\ } & \mathrm{Hom}(V,\mathcal{O}(-1))\\
\downarrow & & A & & \downarrow\\
\mathbb{P}(V) & \ni Ker(A) \dashleftarrow & & \dashrightarrow Im(A)\in & \mathbb{P}(W)
\end{array}$$

The skew arrows are defined only on the open subset of Y consisting of matrices of the rank exactly one.

Let $\mathbb{T}$ be the torus of dimension 4 with basis characters s_1, s_2, t_1, t_2. Suppose the weights of $\mathbb{T}$ acing on V are $-t_1, -t_2$ and s_1, s_2 are the weights of the action on W. There are two fixed points of the torus action at each resolution. The associated torus weights at the fixed points are:

$$\begin{aligned} &\text{Fixed point weights for } X_1: \quad \{t_1 - t_2, s_1 + t_2, s_2 + t_2\}, \ \ \{t_2 - t_1, s_1 + t_1, s_2 + t_1\},\\ &\text{Fixed point weights for } X_2: \quad \{s_2 - s_1, s_1 + t_1, s_1 + t_2\}, \ \{s_1 - s_2, s_2 + t_1, s_2 + t_2\}. \end{aligned}$$

Localization for torus action (Theorem 2.1) allows to compute the equivariant push forward to $H^*_{\mathbb{T}}(\mathbb{C}^4)$:

- Resolution X_1:

$$\mathcal{S}_1 = \frac{\psi_{1*}\mathfrak{c}_f(X_1)}{e(\mathbb{C}^4)} = \frac{1}{f(t_1-t_2)f(s_1+t_2)f(s_2+t_2)} + \frac{1}{f(s_1+t_1)f(s_2+t_1)f(t_2-t_1)}\,.$$

- Resolution X_2:

$$\mathcal{S}_2 = \frac{\psi_{2*}\mathfrak{c}_f(X_2)}{e(\mathbb{C}^4)} = \frac{1}{f(s_2-s_1)f(s_1+t_1)f(s_1+t_2)} + \frac{1}{f(s_1-s_2)f(s_2+t_1)f(s_2+t_2)}\,.$$

We write the difference of the two localization expressions $\mathcal{S}_1 - \mathcal{S}_2$ as a quotient, i.e. we find the common denominator. The numerator is equal to

$$\begin{aligned} \mathcal{R} = f(s_1-s_2)f(s_2-s_1)f(t_1-t_2)f(s_1+t_2)f(s_2+t_2)&\\ + f(s_1-s_2)f(s_2-s_1)f(t_2-t_1)f(s_1+t_1)f(s_2+t_1)&\\ - f(s_2-s_1)f(t_1-t_2)f(t_2-t_1)f(s_1+t_1)f(s_1+t_2)&\\ - f(s_1-s_2)f(t_1-t_2)f(t_2-t_1)f(s_2+t_1)f(s_2+t_2)&\,. \end{aligned}$$

We have assumed that $f(x)$ has the form:

$$f(x) = x\exp\Big(\sum_{i=1}^{\infty} b_i x^i\Big)$$

The relation $\mathcal{R} = 0$ is preserved under the scaling substitution $f(x) \mapsto e^{\lambda x} f(x)$, thus we can assume that $b_1 = 0$, hence $f''(0) = 0$. Let us define the differential relation

$$\mathcal{DR} = \frac{d^4}{ds_1^2 ds_2^2}\mathcal{R}\Big|_{s_1=s_2=t_1=0,\ t_2=t} = 0\,.$$

Having $f(0) = 0$, $f'(0) = 1$, and $f''(0) = 0$, the expression simplifies to:

$$\mathcal{DR} = 4f(-t)\left(2f'(t)^2 - f^{(3)}(0)f(t)^2 - f(t)f''(t)\right) + 8f(t)\,. \tag{11}$$

Substituting the exponential form of $f(x) = x\,e^{\mathbf{r}(x)}$, with $\mathbf{r}(0) = \mathbf{r}'(0) = 0$, we expand in powers of t and extract coefficients. Let d_j denote the coefficient of t^j in the differential expression $\mathcal{DR}$. Calculation shows that $d_j = 0$ for $j < 6$ and the coefficient of d_j depends on b_{j-1} linearly for $j \geq 6$, for example:

$$\begin{aligned} d_6 &= -8\left(6b_2b_3 - 5b_5\right)\,,\\ d_7 &= -\tfrac{4}{3}\left(8b_2^3 + 24b_4b_2 + 27b_3^2 - 42b_6\right)\,. \end{aligned}$$

We find the coefficients of b_j in d_{j+1}:

$$40, 56, 112, 144, 216, 264, 352, 416, 520, 600, 720, 816, 952, 1064, 1216, \ldots$$

Taking second differences of the above sequence we obtain the sequence

$$40, -24, 40, -24, 40, -24, 40, -24, \ldots$$

This indicates that the coefficients of the original sequence split into two quadratic subsequences. This can be easily proven from the form of the differential equation. Alternation appears since $f(-t)$ is present in (11), so even and odd coefficients b_i should be treated separately. None of the coefficient vanishes, therefore one can compute b_i for $i > 4$. The formula depends polynomialy on b_2, b_3 and b_4. Together with b_1 we have four degrees of freedom. First few resulting formulas are given below:

$$\begin{aligned}
b_5 &= \tfrac{6}{5} b_2 b_3 \\
b_6 &= \tfrac{1}{42}\left(8b_2^3 + 24b_4b_2 + 27b_3^2\right) \\
b_7 &= \tfrac{6}{7} b_3\left(b_2^2 + b_4\right) \\
b_8 &= \tfrac{1}{21}\left(b_2^4 + 10b_4b_2^2 + 27b_3^2b_2 + 7b_4^2\right) \\
b_9 &= \tfrac{1}{7} b_3\left(4b_2^3 + 12b_4b_2 + 3b_3^2\right) \\
b_{10} &= \tfrac{2}{1155}\left(28b_2^5 + 160b_4b_2^3 + 945b_3^2b_2^2 + 340b_4^2b_2 + 540b_3^2b_4\right) \\
b_{11} &= \tfrac{4}{77} b_3\left(8b_2^4 + 38b_4b_2^2 + 27b_3^2b_2 + 14b_4^2\right) \\
&\vdots
\end{aligned}$$

We know that

$$f(x) = \frac{\theta_\tau(x+h)\,\theta'_\tau(0)}{\theta_\tau(x)\,\theta_\tau(h)}$$

satisfies the above constraints. Modifying $f(x)$ by the factor $e^{\alpha x}$ and rescaling x, we achieve three degrees of freedom, keeping τ fixed.

Theorem 10.1. *The characteristic class $\mathfrak{c}_f(Y) = \psi_* \mathfrak{c}_f(X)$ does not depend on the crepant resolution $\psi : X \to Y$ if and only if*

$$f(x) = e^{\alpha x} \frac{\theta_\tau(\nu x + h)\theta'_\tau(0)}{\nu\theta_\tau(\nu x)\theta_\tau(h)}$$

for $\alpha, \nu, h, \tau \in \mathbb{C}$, $\nu \neq 0$, $\mathrm{im}(\tau) > 0$ or $f(x)$ is a limit of functions of the above form.

Of course the part "if" follows from the properties of the elliptic characteristic class proved in [6].

Remark 10.2. Comparing with [20] we use a different expansion of $f(x)$. The exponential expansion is convenient, because it allows easily to get rid of the parameter b_1, which does not enter into the relation.

11. Symplectic singularities and Grassmannian flops

According to [5] a symplectic singular variety Y is a normal variety whose regular part is a symplectic manifold and there exists a resolution $\psi : X \to Y$ and $\psi^*(\omega_{Y_{\mathrm{reg}}})$ extends over X to a symplectic form. The variety X together wit the map to Y is called a symplectic resolution. The examples of symplectic singular manifolds are closures of nilpotent orbits in a semisimple Lie algebra, as described below.

Let us concentrate on the nilpotent orbits of the type A, i.e. the conjugacy classes of nilpotent matrices. We assume that the nilpotency order is ≤ 2. Consider the closure of the orbit of a matrix of rank k with square zero. There are two well known symplectic resolusion, see [12] for the classifications of the symplectic resolutions for general case.

Here both resolutions are the cotangent spaces of Grassmannians:

$$\begin{array}{ccccc}
X_1 & & Y & & X_2 \\
T^*\mathrm{Grass}_{n-k}(\mathbb{C}^n) & \xrightarrow{\psi_1} & \left\{\begin{matrix} A:\mathbb{C}^n\to\mathbb{C}^n \\ A^2=0,\ rkA\le k\end{matrix}\right\} & \xleftarrow{\psi_2} & T^*\mathrm{Grass}_k(\mathbb{C}^n) \\
\downarrow & & A & & \downarrow \\
\mathrm{Grass}_{n-k}(\mathbb{C}^n) & \ni Ker(A) & \swarrow\ \ \searrow & Im(A)\in & \mathrm{Grass}_k(\mathbb{C}^n)
\end{array}$$

Let $k=1$. Both resolutions of the nilpotent cone are the isomorphic to the cotangent bundles of projective spaces. We consider the multiplicative class $\mathfrak{c}_f$ and compute the push-forward of $\mathfrak{c}_f(T^*\mathbb{P}^{n-1})$ using resolutions X_1 and X_2.

Let $\mathbb{T}=\mathbb{C}^*\times(\mathbb{C}^*)^n$ with coordinates s and x_i for $i=1,2,\dots,n$. The first factor acts by the scalar multiplication, and the second factor acts by conjugation. For each n, define the following sums:

$$\mathcal{S}_1(n)=\frac{\psi_{1*}\mathfrak{c}_f(X_1)}{e(\mathrm{Hom}(\mathbb{C}^n,\mathbb{C}^n))}=\sum_{p=1}^{n}\prod_{\substack{i=1\\ i\neq p}}^{n}\frac{1}{f(x_i-x_p)\,f(x_p-x_i+s)},$$

$$\mathcal{S}_2(n)=\frac{\psi_{2*}\mathfrak{c}_f(X_2)}{e(\mathrm{Hom}(\mathbb{C}^n,\mathbb{C}^n))}=\sum_{p=1}^{n}\prod_{\substack{i=1\\ i\neq p}}^{n}\frac{1}{f(x_p-x_i)\,f(x_i-x_p+s)}.$$

These expressions are equal to the push forwards of the classes $\mathfrak{c}_f(X_i)$ divided by the Euler class of $\mathrm{Hom}(\mathbb{C}^n,\mathbb{C}^n)$. For example for $n=3$:

$$\begin{aligned}\mathcal{S}_1(3)=&\frac{1}{f(s+x_1-x_2)\,f(-x_1+x_2)\,f(s+x_1-x_3)\,f(-x_1+x_3)}\\ &+\frac{1}{f(x_1-x_2)\,f(s-x_1+x_2)\,f(s+x_2-x_3)\,f(-x_2+x_3)}\\ &+\frac{1}{f(x_1-x_3)\,f(x_2-x_3)\,f(s-x_1+x_3)\,f(s-x_2+x_3)}.\end{aligned}$$

To obtain $\mathcal{S}_2(3)$ one has to change the sign of all x-variables. We are looking for the functions f for which $\mathcal{S}_1(3)=\mathcal{S}_2(3)$. Let $\mathcal{R}(3)$ be the numerator of the difference $\mathcal{S}_1(3)-\mathcal{S}_2(3)$ brought to a common denominator. To study the vanishing of the difference $\mathcal{R}(3)$, we analyze the coefficient of the expansion under the assumption that $f(x)$ satisfies:

$$f(0)=0,\quad f'(0)=1,\quad f''(0)=0.$$

We can assume that $f''(0)=0$ because the condition $\mathcal{R}(3)$ does not change after replacing $f(x)$ by $e^{\lambda x}f(x)$. Define the differential expression:

$$\mathcal{DR}_3=\frac{d^9}{dx_1^4dx_2^5}\mathcal{R}_3\Big|_{x_1=x_2=x_3=0},$$

$$\begin{aligned}\mathcal{DR}_3=40\Big(&360f'(s)^3f''(s)\\ &-60f(s)f'(s)\left(3f''(s)^2+2f^{(3)}(0)f'(s)^2+4f^{(3)}(s)f'(s)\right)\\ &+60f(s)^2\left(f^{(4)}(s)f'(s)+f^{(3)}(s)f''(s)+f^{(3)}(0)f'(s)f''(s)\right)\\ &+f(s)^3\left(f^{(5)}(0)f'(s)-f^{(5)}(s)\right)\\ &+f(s)^4\left(f^{(6)}(0)-5f^{(3)}(0)f^{(4)}(0)\right)\Big).\end{aligned}\tag{12}$$

Substituting $f(x)$ by $x\exp\left(\sum_{i\geq 2} b_i x^i\right)$, we compute the Taylor coefficients of $\mathcal{DR}_3$ in powers of s. The coefficient of s^j are denoted d_j, for example:

$$d_7 = -57600\left(b_2^4 + 2b_2^2 b_4 - b_4^2 - 6b_2 b_6 + 3b_8\right),$$
$$d_8 = -28800\left(12b_3 b_2^3 + 10b_5 b_2^2 + 12b_3 b_4 b_2 - 42b_7 b_2 - 10b_4 b_5 - 18b_3 b_6 + 27b_9\right)$$

and so on. The coefficient d_j is linear in the variable b_{j+1}, and d_j does not depend on b_k for $k > j+1$. We compute the sequence of the coefficient of b_{j+1} in d_j for $j \geq 7$. Since the number differentiations in the monomials of (12) is at most 5 (there are monomials $(f')f''$, $f'(f'')^2$, ...) the coefficient of b_{j+1} in d_j is a polynomial of degree ≤ 5. The first few coefficients are equal:

$$-172800, -777600, -2217600, -5068800, -10108800, -18345600, -31046400, -49766400, \dots$$

Having initial values we find that the coefficient of b_{j+1} is equal to

$$-240(j-6)(j-5)(j-2)(j+1)(j+2),$$

hence it does not vanish for $j > 6$. Therefore the unknown variables b_j for $j > 7$ is determined by b_i for $i = 2, 3, \dots, 7$.

A similar analysis for $n = 4$ yields a higher-order differential equation. Define

$$\mathcal{DR}_4 = \frac{d^{15}}{dx_1^5 dx_2^5 dx_3^5}\mathcal{R}_4\Big|_{x_1=x_2=x_3=x_4=0}.$$

This leads to a complicated expression involving derivatives of $f(x)$ up to sixth order, and again leads to relations among the coefficients b_i in the expansion of $f(x)$. The first nonvanishing coefficients of d_j is

$$d_{13} = -27648000\left(128b_2^5 + 160b_4 b_2^3 - 780b_6 b_2^2 - 240b_4^2 b_2 + 600b_8 b_2 + 240b_4 b_6 - 165b_{10}\right).$$

Having only the relations for $n = 4$ we would be able only to determine the variables starting from b_{10} as polynomials in b_i, $i = 2, 3, \dots, 9$. We claim that together with the relations deduced from $n = 3$ all the coefficients b_j can be expressed as polynomials in b_2, b_3, b_4. This is checked by a direct computation in Wolfram Mathematica. See §12.

We have analyzed two different Grassmannian flops in type A via torus localization and found two differential equation. The differential equations produce polynomial constraints among the coefficients of the generating power series $f(x)$. Having two independent differential equations involving the function f we find that there is only four degrees of freedom.

Theorem 11.1. *Suppose that $\mathfrak{c}_f$ is invariant with respect to Grassmannian flops, then f is of the form as in Theorem 10.1.*

Remark 11.2. An examination of the nilpotent orbit of 5×5 matrices of rank at most 2 and square zero leads to the same constraints on the coefficients as in the case of 3×3 matrices of rank one. There are many other tractable cases that one could compute, but we have not studied them systematically, since just two simple Grassmannian flops already determine the elliptic genus.

12. Codes

This script is written in Wolfram Mathematica. It contains all the calculations discussed in the paper. The source: `https://www.mimuw.edu.pl/~aweber/4paths`

```
md=20; (* maximal defree *)
expansion=Normal[Series[u Exp[Sum[a[i]u^i,{i,4,md+5,2}]],{u,0,md+5}]];
ff[x_]:=expansion/.u->x
FF[x_,y_]:=f[x+y]/(f[x]f[y])
Print["Blow-up relation"]
LHS=F[x1,b1+b2] F[x2-x1,b2]+F[x2,b1+b2] F[x1-x2,b1];
RHS=F[x1,b1] F[x2,b2];
Rel0=Numerator[Factor[LHS-RHS/.F->FF]]
Print["if f''[0]=0, then f is an odd function"]
inic={f[0]->0,f'[0]->1,f''[0]->0};
Factor[D[Rel0,{x2,1},{b1,1},{b2,1}]/.{x2->0,b1->0,b2->0,x1->x}/.inic]==0
Print["relation, assuminf that f is antisymmetric"]
Rel=Factor[Rel0/f[x1-x2]/.f[x2-x1]->-f[x1-x2]]
Print["differential relation for odd function"]
inic={f[0]->0,f'[0]->1,f''[0]->0,f'''[0]->0};
asym={f[-x]->-f[x],f'[-x]->f'[x],f''[-x]->-f''[x],f'''[-x]->f'''[x],f
    ''''[-x]->-f''''[x]};
DR=Factor[D[Rel,{x2,2},{b1,2},{b2,2}]/.{x2->0,b1->0,b2->0,x1->x}/.inic/.
    asym]
Print["the sequence of relations, d[j] depends linearly on a[j+2]"]
Do[d[j]=Factor[Coefficient[DR/.f->ff,x,j]],{j,0,md,2}]
Do[Print["d[",j,"]=",d[j]],{j,0,16,2}]
Print["coefficients obey polynomial pattern -2(j-4)(1+j)(2+j)(3+j)"]
Table[-2(j-4)(1+j)(2+j)(3+j)==Coefficient[d[j],a[j+2]],{j,6,md-2,2}]
Print["expressions for a[j], j>8"]
p={};Do[p=Union[p,Solve[0==d[i]/.p,a[i+2]][[1]]],{i,6,md,2}];
Column[Expand[p]]

Print["a simpler form of the equation"]
inic={v[0]->0,v'[0]->0,v''[0]->0};
Expand[Solve[0==DR/.f->Function[x,x  E^(-2 v[x])]/.inic,v''''[x]]]

Print["Braid relation"]
LHS=F[z2-z1,m3-m2] F[z3-z2,m3-m1] F[z2-z1,m2-m1]+F[z1-z2,h] F[z3-z1,m3-m1]
     F[z2-z1,h];
RHS=F[z3-z2,m2-m1] F[z2-z1,m3-m1] F[z3-z2,m3-m2]+F[z2-z3,h] F[z3-z1,m3-m1]
     F[z3-z2,h];
Rel0=Numerator[Factor[LHS-RHS/.F->FF/.m1->0/.z1->0]]
Print["if f''[0]=0,then f is an odd function"]
inic={f[0]->0,f'[0]->1,f''[0]->0};
Factor[D[Rel0,{m2,1},{m3,1},{z2,3},{h,1}]/.{m2->0,m3->0,z2->0,h->0}/.inic
    ]==0
Print["relation,assuminf that f is antisymmetric"]
Rel=Factor[Rel0/(f[z2]f[z2-z3])/.{f[-z2]->-f[z2],f[z3-z2]->-f[z2-z3]}]
Print["differential relation for odd function"]
inic={f[0]->0,f'[0]->1,f''[0]->0,f'''[0]->0};
asym={f[-x]->-f[x],f'[-x]->f'[x],f''[-x]->-f''[x],f'''[-x]->f'''[x],f
    ''''[-x]->-f''''[x]};
DBr=Factor[D[Rel,{m2,1},{m3,2},{z2,2},{h,4}]/.{m2->0,m3->0,z2->0,z3->x,h
    ->0}/.inic/.asym]
Print["new differential equation differs by an invertible factor"]
Factor[DBr/DR]

Print["Atiyah Flop"]
expansion=Normal[Series[u Exp[Sum[b[i]u^i,{i,2,md+14}]],{u,0,md+14}]];
ff[x_]:=expansion/.u->x
```

```
Print["the relation"]
SA1=1/(f[t1-t2] f[s1+t2] f[s2+t2])+1/(f[s1+t1] f[s2+t1] f[-t1+t2]);
SA2=1/(f[-s1+s2] f[s1+t1] f[s1+t2])+1/(f[s1-s2] f[s2+t1] f[s2+t2]);
RA=Numerator[Factor[SA1-SA2]]
Print["differential equation assuminf f(0)=0, f'(0)=1, f''(0)=0"]
inic={f[0]->0,f'[0]->1,f''[0]->0};
DR=Factor[D[RA,{s1,2},{s2,2}]/.{s1->0,s2->0,t1->0,t2->t}/.inic];
Collect[DR,f[-t],Factor]
Print["the sequence of relations: d[j] depends linearly on b[j-1]"]
Do[d[j]=Factor[Coefficient[DR/.f->ff,t,j]],{j,0,md}]
Do[Print["d[",j,"]=",d[j]],{j,0,10}]
Print["coefficient obey a polynomial pattern, 4(j-4)(j-1), j even "]
Table[Coefficient[d[j],b[j-1]]==4(j-4)(j-1),{j,6,md,2}]
Print["coefficient obey a polynomial pattern, 4(j-4)j, j odd "]
Table[Coefficient[d[j],b[j-1]]==4(j-5)j,{j,5,md,2}]
Print["expressions for a[j] for j>4"]
p={};Do[p=Union[p,Solve[0==d[i]/.p,b[i-1]][[1]]],{i,6,md}];
Column[Expand[p]]

Print["frassmannian flop"]
S1[n_]:=Sum[Product[1/(f[x[i]-x[p]]f[x[p]-x[i]+s]),{i,Complement[Range[n],{p}]}],{p,1,n}]
S2[n_]:=Sum[Product[1/(f[x[p]-x[i]]f[x[i]-x[p]+s]),{i,Complement[Range[n],{p}]}],{p,1,n}]
R[n_]:=Numerator[Factor[S1[n]-S2[n]]]
inic={f[0]->0,f'[0]->1,f''[0]->0};

Print["differential equation for the nilpotent cone, n=3"]
DR3=Factor[D[Factor[D[R[3]/.x[1]->0,{x[2],5}]/.x[2]->0]/.inic,{x[3],4}]/.x[3]->0/.inic];
Collect[DR3/40,f[s],Factor]
Print["the sequence of relations, d[j] depends linearly on b[j-1]"]
Do[d[j]=Factor[SeriesCoefficient[DR3/.f->ff,{s,0,j}]],{j,0,md}]
Do[Print["d[",j,"]=",d[j]],{j,0,16}]
Print["the polynomial pattern -240(j-6)(j-5)(j-2)(j+1)(j+2)"]
Table[Coefficient[d[j],b[j+1]]==-240(j-6)(j-5)(j-2)(j+1)(j+2),{j,7,md}]
Print["expressions for b[j] for j>7"]
p3={};Do[p3=Union[p3,Solve[0==d[i]/.p3,b[i+1]][[1]]],{i,7,md}];
Column[Expand[p3]]

Print["differential equation for the nilpotent cone, n=4"]
DR4=Factor[D[R[4]/.x[1]->0,{x[2],5}]/.x[2]->0/.inic];
DR4=Factor[D[DR4,{x[3],5}]/.x[3]->0/.inic];
DR4=Collect[D[DR4,{x[4],5}]/.x[4]->0/.inic,f[s],Factor];
Collect[DR4/144000,f[s]]
Print["the sequence of relations, d[j] depends linearly on b[j-3]"]
Do[d[j]=Factor[SeriesCoefficient[DR4/.f->ff,{s,0,j}]],{j,0,md}]
Do[Print["d[",j,"]=",d[j]],{j,0,16}]
Print["polynomial pattern 144000(j-12)(j-11)(j-10)(j-3)(j-2)(7j-43)"]
Table[Coefficient[d[j],b[j-3]]==144000(j-12)(j-11)(j-10)(j-3)(j-2)(7j-43),{j,13,md}]
Print["expressions for b[j] j>9"]
p4={};Do[p4=Union[p4,Solve[0==d[i]/.p4,b[i-3]][[1]]],{i,13,md}];
Column[Expand[p4]]

Print["combined relations"]
```

```
diff=Factor[(Table[b[i],{i,10,13}]/.p3)-Table[b[i],{i,10,13}]/.p4/.p3];
Solve[diff=={0,0,0,0},Table[b[i],{i,5,7}]]
```

The code was intentionally written to be as simple as possible. We did not implement optimizations for computational efficiency. Still the code performs well for a reasonable range of degrees.

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