EKT&EC — Problem List 3: to be solved 18.03

Problem 1 (*More about structure of G-spaces.*). Let X be a smooth manifold, on which a Lie group G acts smoothly. For a subgroup $H < G$ let

$$X_{(H)} = \Big\{ x \in X \ : \ \exists g \in G \ \ gG_x g^{-1} = H \Big\}$$

i.e. the set of points whose stabilizers are conjugate to H. Show that $X_{(H)} \to X_{(H)}/G$ is a locally trivial fiber bundle.)

(It is associated to a principal bundle with the structure group $N(H)/H$.)

Problem 2 Find a decomposition

$$\begin{pmatrix} t^2 & t^3+t^4 \\ 0 & t^5 \end{pmatrix} = A_+ D A_-$$

such that $A_+ \in \mathrm{GL}_n(\mathbb{C}[t])$, $A_- \in \mathrm{GL}_n(\mathbb{C}[t^{-1}])$ and $D \in \mathrm{GL}_n(\mathbb{C}[t,t^{-1}])$ is a diagonal matrix.

See Birkhoff factorization, e.g. the algorithm from Wikipedia.

Problem 3 Let G be a compact group. Prove the cancellation property:

$$V_1 \oplus W \simeq V_2 \oplus W \quad \Longrightarrow \quad V_1 \simeq V_2 \, .$$

Give an example of a noncompact group which does not have the cancellation property.

Problem 4 Suppose $P \in \mathrm{End}(\mathbb{C}^n)$. Let

$$\tfrac{1}{2\pi i} \int_\gamma (zI - P)^{-1} dz \, ,$$

where γ is the circle around 1 with radius $1/2$. It is assumed, that P has no eigenvalues in that circle. Show $\Phi(P)$ is an idempotent.

How to describe the kernel and the image of $\Phi(P)$?

This operation allows to convert any P, which is sufficiently close to an idempotent into an actual idempotent. The map Φ is invariant with respect to $\mathrm{GL}_n(\mathbb{C})$ acting on $\mathrm{End}(\mathbb{C}^n)$.

Problem 5 Show, that we have

$$[\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)] = [\mathcal{O}_{\mathbb{P}^1}(a') \oplus \mathcal{O}_{\mathbb{P}^1}(b')] \in K_{loc.free}(\mathbb{P}^1)$$

if and only if $a + b = a' + b'$. Conclude that

$$K_{loc.free}(\mathbb{P}^1) \to K(\mathbb{P}^1)$$

is an isomorphism. Here $K_{loc.free}(\mathbb{P}^1)$ stands for the K-theory of locally free sheaves of $\mathcal{O}_{\mathbb{P}^1}$-modules, i.e. the complex holomorphic vector bundles.

Problem 6 Suppose S^1 acts on S^2 by rotations around a fixed axis. Compute $K_{S^1}(S^2)$.