Equivariant K-theory and Elliptic cohomology

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I will assume that the participants of the course are well familiar with the classical cohomology theory.

Introduction

Summary of Cohomology Theories on the category of Compact Spaces

0.1 Let $\mathbf{Top}_\star$ denote the category of compact topological spaces with a distinguished point and continuous maps. We assume that the spaces are homeomorphic to polyhedra (or CW-complexes). By $\star$ we denote the one point space or the distinguished point in the topological space X.

0.2 Depending on the circumstances we assume that the spaces are "decent". We will discuss technical issues later. It is most convenient to a assume that the spaces we consider are CW-complexes.

0.3 Let $\mathbf{hTop}_\star$ be the homotopy category.

A short formulation of cohomology axioms

0.4 A cohomology theory consists of:

- A sequence of contravariant functors

$$\widetilde{H}^n : \mathbf{hTop}_\star \to \mathbf{Ab}, \quad n \in \mathbb{Z},$$

- Natural transformations (connecting homomorphisms)

$$\delta : \widetilde{H}^n(A) \to \widetilde{H}^{n+1}(X/A)$$

  for each pair (X, A) with $A \subseteq X$ closed.

0.5 In **standard cohomology theory** the following axioms are satisfied

- (i). Exactness Axiom

$$\cdots \to \widetilde{H}^n(X/A) \to \widetilde{H}^n(X) \to \widetilde{H}^n(A) \xrightarrow{\delta} \widetilde{H}^{n+1}(X/A) \to \cdots$$

- (ii). Dimension Axiom

$$\widetilde{H}^n(S^0) = \begin{cases} R & \text{if } n = 0, \qquad \text{a fixed abelian group/ring (the coefficients)} \\ 0 & \text{if } n \neq 0. \end{cases}$$

Here S^0 denotes the 0-dimensional sphere, i.e. two point space, one of the points is distinguished.

0.6 This data determines the theory.

0.7 We write:

$H^n(X) = \widetilde{H}^n(X^+)$, where $X^+ = X \sqcup \star$,

$H^n(X, A) = \widetilde{H}^n(X/A)$ for $A \subset X$ closed (cofibration),

$H^*(X) = \bigoplus_{n\in\mathbb{Z}} H^n(X)$,

If we want to stress the coefficients (e.g. in $\mathbb{Z}$) we write $H^*(X; \mathbb{Z})$.

0.8 We deduce the classical set of Eilenberg-Steenrod axioms

- exactness axiom
- excision axiom
- dimension axiom
- for noncompact spaces one has to assume an additional axiom

$$H^*(\bigsqcup_\alpha X_\alpha) = \prod_\alpha H^*(X_\alpha)\,.$$

0.9 Let

$$A \vee B = A \times \star \;\cup\; \star \times B$$

and

$$A \wedge B = A \times B / A \vee B\,.$$

- Show that $S^1 \wedge X$ is homotopy equivalent to $\Sigma X = [0,1] \times X/\sim$

$$(0,x) \sim (0,y)\,, \qquad (1,x) \sim (1,y)$$

0.10 **Generalized cohomology theory**: There is only one axiom (i). We reject the dimension axiom.

0.11 Exactness Axiom:

$$\cdots \to \widetilde{h}^n(X/A) \to \widetilde{h}^n(X) \to \widetilde{h}^n(A) \xrightarrow{\delta} \widetilde{h}^{n+1}(X/A) \to \cdots$$

- No dimension axiom !

0.12 The knowledge of $\widetilde{h}^*(S^0)$ does not determine the theory: If $\widetilde{h}^*(S^0)$ is torsion free then

$$\widetilde{k}^*(X) := \widetilde{H}^*(X;\mathbb{Z}) \otimes \widetilde{h}^*(S^0)$$

is another cohomology theory. The theories h and k have the same value for $X = S^0$, but in general they are not isomorphic.

K-theory:	$\widetilde{K}^*(-)$	$\not\simeq H^*(-) \otimes \widetilde{K}^*(S^0) \simeq \bigoplus_{k\in\mathbb{Z}} H^{*+2k}(-)\,,$
Unitary cobordism:	$\widetilde{U}^*(-)$	$\not\simeq \widetilde{H}^*(-) \otimes \widetilde{U}^*(S^0)\,.$

0.13 Exercise: show that

- $\widetilde{h}^*(\star) = 0$
- $\widetilde{h}^*(A \wedge B) \simeq \widetilde{h}^*(A) \oplus \widetilde{h}^*(B)$
- $\widetilde{h}^k(X) \simeq \widetilde{h}^{k+1}(S^1 \wedge X)$.

0.14 Cohomology theories are represented by spectra: (see Brown's representability theorem, [R. Switzer, *Algebraic topology - homotopy and homology*, Classics in Mathematics, Berlin, New York: Springer-Verlag, pp. 152–157]).

0.15 ***G*-spaces**
Let G be a compact group, $G - \mathbf{hTop}_\star$ the category of compact G-spaces with a distinguished point, which is preserved by G. Additionally one has to exclude topological pathologies. The notion of G-CW-complexes is most convenient . For spaces like manifolds, algebraic varieties with smooth/algebraic group actions the topological complications do not appear.

0.16 Basic equivariant construction:

- For an orthogonal representation $H \to Aut(V)$ let $D(V)$ denote the unit ball in V and $S(V) = \partial D(V)$ the unit sphere.
- For $H \subset G$ and a H-space Y let

$$G \overset{H}{\times} Y = G \times Y/\sim, \qquad (gh, y) \sim (g, hy) \text{ for } h \in H$$

We have for a G-space Z

$$Map_H(X,Z) = Map_G(G \overset{H}{\times} X, Z)\,.$$

- Applying $\overset{H}{\times}$ we construct a building block, an ,,equivariant cell" – a tubular thickening $G \overset{H}{\times} D(V)$ of the orbit G/H, whose boundary $G \overset{H}{\times} S(V)$ is a sphere bundle over G/H.

0.17 A G-space X admits a G-CW decomposition if there is a filtration

$$\emptyset = X_0 \subset X_1 \subset \cdots \subset X_n = X$$

such that for each k there are given a subgroup H_k, its orthogonal representation $H_k \to GL(V_k)$ and a G-homeomorphism

$$X_k \simeq X_{k-1} \cup_{f_k} (G \overset{H_k}{\times} D(V_k))$$

where $f_k : G \overset{H_k}{\times} S(V_k) \to X_{k-1}$ is the gluing data, ,,the characteristic map".

- Further, we can assume (possibly passing to a finer decomposition) that the action of H on V is trivial, thus the cells are of the form $G/H \times D(V)$.

- For a proof that a smooth G-manifold with a smooth G-action of a Lie group admits a decomposition into G-cells (in the strong sense) see [Sören Illman, *The Equivariant Triangulation Theorem for Actions of Compact Lie Groups.* Mathematische Annalen (1983) Volume: 262, page 487-502]

0.18 The assumption that there is a distinguished fixed point is made for convenience. If the action of G on X has no fixed point, we can artificially add a point considering $X^+ = X \sqcup \star$.

0.19 To dive dipper into the structure of G-spaces, see *slice theorem* [Bre72, §II.5].

- Example: equivariant tubular neighbourhood.
- Peculiarities of algebraic tori acting on algebraic varieties, [Car02, §4.1].

Generalized equivariant cohomology theory

0.20 Exactness Axiom

$$\cdots \to \widetilde{h}^n(X/A) \to \widetilde{h}^n(X) \to \widetilde{h}^n(A) \xrightarrow{\delta} \widetilde{h}^{n+1}(X/A) \to \cdots$$

0.21 Suspension is more involving, and it is included into axioms:

- Let $V \simeq \mathbb{R}^d$ be a linear representation of G , and $S_V = V \sqcup \{\infty\}$, i.e. one point compatification of V, which is a sphere with G action. It has at least two points: 0 and ∞.
- Let $|V|$ denote the real dimension of V.
- There is given a family of **compatible** isomorphisms, see [tD71]

$$\widetilde{h}^n(X) \simeq \widetilde{h}^{n+|V|}(S_V \wedge X)\,.$$

(Easier to say $h^n(X) \simeq h^{n+|V|}(S_V \times X, \{\infty\} \times X)$.)

0.22 Compatibilities of the equivariant suspensions:

- Behavior with respect to taking the direct sum of representations. There is given a family of isomorphisms

$$\widetilde{h}^*(S_V \wedge S_W \wedge X) \simeq \widetilde{h}^*(S_W \wedge S_V \wedge X) \simeq \widetilde{h}^*(S_{V\oplus W} \wedge X)\,,$$

which satisfy a (easy to guess) coherences involving signs.

- Commutation of δ with $S_V\wedge$:

We leave details for later.

0.23 Transformation of theories. Although coefficients of the theory do not determine it, but in the presence of transformation of theories one can conclude the isomorphism:

- If $k^* \to h^*$ is a transformation of generalized equivariant cohomology theories and suppose that for every orbit G/H the induced map

$$(\spadesuit) \qquad k^*(G/H) \xrightarrow{\simeq} h^*(G/H)$$

is an isomorphism. Then for any G-space X

$$k^*(X) \xrightarrow{\simeq} h^*(X)\,.$$

- Enough to assume (♠) for orbits appearing in X
- That is so because here the building blocks is not the point $\star$, but the orbits G/H.

Instead of developing an abstract theory let us study in detail K-theory, [Seg68].

1 Vector bundles

See [HJJS08].

1.1 (Non-equivariant) vector bundles and constructions on bundles – recollection.
We fix $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ (possibly $\mathbb{H}$), but preferably $\mathbb{C}$.
Definition: a vector bundle over $\mathbb{K}$ is a map $p : E \to X$ together with continuous operations

$$+ : E \times E \to E, \quad \cdot : \mathbb{K} \times E \to E, \quad 0 : X \to E\,,$$

such that:

- ◦ $p : E \to X$ is a local trivial fibration,
- ◦ the maps $+$ and $\cdot$ preserve the fibers of p, the map 0 is a section of p,
- ◦ each fiber $E_x = p^{-1}(x)$ is a vector space w/r to $+$ and $\cdot$.

1.2 Examples

- Trivial bundle $X \times \mathbb{K}^n$. If $n = 1$ such bundle is often denoted by $1\!\!1_X$ or θ_X.
- Bundles known from diffrerential geometry:

◦ If X is a manifold over $\mathbb{R}$ or $\mathbb{C}$, then we have the tangent vector bundle TX, cotangent vector bundle T^*X, also denoted by Ω^1_X, but this can be confused with the sheaf of sections.

◦ Exterior powers of the cotangent bundle — do not confuse with the shaves of forms. (But in practice often denoted by the same symbol Ω^p_X.)

◦ If $Y \subset X$ is a submanifold: normal bundle $\nu_{Y/X} = (TX|_Y)/TY$, conormal bundle $\nu^*_{Y/X} = ker(T^*X|_Y \to T^*Y)$,

◦ If $f : X \to Y$ is a submersion, then we have the relative tangent bundle, $Tf \subset TX$, consisting of vectors tangent to the fibers.

- Tautological bundles: Let V be a vector space, $\dim V = n < \infty$ and let $\mathrm{Gr}_k(V)$ be the Grassmann manifold parametrizing k-dimensional subspaces. We have

◦ $S = \{(W, v) \in \mathrm{Gr}_k(V) \times V \;:\; v \in W\}$ the tautological bundle, it is a subbundle of the trivial bundle $\underline{V} = \mathrm{Gr}_k(V) \times V$.

◦ $Q = \underline{V}/S$ - the quotient bundle.

◦ As topological bundles $\underline{V} \simeq S \oplus Q$, (an isomorphism determined e.g. by a choice of a scalar/hermitian product in V). But as holomorphic bundles we have only an exact sequence.

◦ In particular for $k = 1$, $Gr_1(V) = \mathbb{P}(V)$ the tautological line bundle is denoted $\mathcal{O}_{\mathbb{P}(V)}(-1)$, and the dual bundle by $\mathcal{O}_{\mathbb{P}(V)}(1)$.

1.3 For the fixed base X the *class* of isomorphism classes of vector bundles of given dimension n is a *set*, moreover (see 2.8)

$$Vect^n : \mathbf{Top}^{op} \to \mathbf{Set}$$

is a functor, which factors through $\mathbf{hTop}^{op}$ (see 2.9).

- We often skip $\mathbb{K}$ in the notation, meaning $\mathbb{K} = \mathbb{C}$.

1.4 Richer categories: smooth, holomorpic, algebraic bundles.

- Associated sheaves of sections.

1.5 Vector bundles can be identified with projective modules over $C(X;\mathbb{K})$.

1.6 Operation on bundles

- Direct sum $E \oplus F = E \times_X F$
- ◦ The set $\bigcup_{n\geq 0} Vect^n(X)$ is a semigroup.
- Tensor product — this demands use of defining cocyles
- ◦ On $\bigcup_{n\geq 0} Vect^n(X)$ we have a structure of a semiring
- Hom-bundle, in particular the dual bundle
- Any functor e.g. symmetric and exterior powers,

1.7 For a subbundle $F \subset E$ we have the quotient E/F

1.8 In general a morphism of bundles $f : E \to F$ might have nonconstant rank, so kernel and cokernel is not defined unless we pass to sheaves (assuming some richer structure, e.g. holomorphic).

1.9 Equivariant vector bundles

- We assume that G acts on E and X and the map p is G equivariant. Moreover, if $g(x) = y$ then the map $g|_{E_x} : E_x \to E_y$ is liner.
- In particular, if x is a fixed point, then E_x is a linear representation of G.

1.10 What are the bundles over G/H? Given a representation $H \to \mathrm{GL}(V)$ define a bundle over G/H

$$G \overset{H}{\times} V = G \times V/\sim\,, \qquad (gh, v) \sim (g, hv)\,.$$

1.11 Theorem: *Let $W \to G/H$ be a vector bundle, $V = W_{eH}$ the fiber over eH, this is an H-representation. Then the natural map*

$$G \overset{H}{\times} V \to W\,, \qquad [g, v] \mapsto gv$$

is an isomorphism of G-bundles.

1.12 Corollary:

$$Vect^n_G(G/H) \simeq Vect^n_H(pt)\,.$$

1.13 Generalization for H-space

$$Vect^n_G(G \overset{H}{\times} X) \simeq Vect^n_H(X)\,.$$

2 Vector bundles and K-theory

2.1 Definition via cocycle

- Given a covering $\{U_i\}_{i\in I}$ and gluing functions $g_{ij} : U_i \cap U_j \to \mathrm{GL}_n(\mathbb{C})$ such that

$$g_{ij}g_{jk} = g_{ik} \quad \text{on } U_i \cap U_j \cap U_k$$

(cocycle condition). We define

$$E = \left(\bigcup_{i\in I}\{i\} \times U_i \times \mathbb{C}^n\right) /\sim$$

$$(i, x, v) \sim (j, x, g_{ij}(x)) \qquad \text{for } x \in U_{ij}\,.$$

- Passing to a subcovering allows to compare cocycles defined for different coverings, like in the Čech cohomology.
- Two cocycles $\{g_{ij}\}$ and $\{g'_{ij}\}$ define isomorphic bundles if (on a finer covering) there exist functions $h_i : U_i \to \mathrm{GL}_n(\mathbb{C})$ such that

$$g'_{ij} = h_i g_{ij} h_j^{-1} \quad \text{on } U_i \cap U_j\,.$$

• In other words
$$Vect^n(X) := H^1(X; \mathcal{C}(X, \mathrm{GL}_n(\mathbb{C})))\,.$$
This is the ,,nonabelian" cohomology with coefficients in the sheaf of groups $\mathcal{C}(X, \mathrm{GL}_n(\mathbb{C}))$ of continuous functions with values in $\mathrm{GL}_n(\mathbb{C})$.

2.2 The same mechanism applies to equivariant bundle. But as a basic "trivial bundles" are the bundles of the form
$$G \overset{H}{\times} (S \times V) \to G \overset{H}{\times} S = U\,,$$
where S is a H-space and V is a representation of H.

• We have to assume that the sets of the covering are as above, hence they are preserved by G and the cocycle is G-equivariant.

• In particular, suppose that the action of G on X is trivial, and the bundle itself is trivial. But it does it does not mean that we can take as a trivializing covering consisting of X.

◦ Example: Suppose $\mathbb{1}^n = E_1 \oplus E_2$ with the action of $G = S^1$ by two different characters on the summands.

2.3 Theorem: *Every bundle is a quotient of a trivial bundle (under standing assumption that X is compact).*

Proof (for the nonequivariant case): Let $n = \mathrm{rk}(E)$. Suppose the bundle E has trivialization on a finite covering $\{U_i\}_{i\in I}$:
$$\varphi_i : U_i \times \mathbb{C}^n \to E|_{U_i}\,.$$
Assume that ψ_i is a partition of unity for the covering $\{U_i\}$. The map
$$X \times \bigoplus_{i\in I} \mathbb{C}^n \to E$$
$$(x, \{v_i\}_{i\in I}) \mapsto \sum_{i\in I} \psi_i(x)\varphi_i(x, v_i)$$
is well defined and surjective. □

2.4 Better point of view: We have a natural map
$$C(X; E) \times X \to E$$
$$(s, x) \mapsto s(x)\,.$$
We pick a finite dimensional vector space $V \subset C(X; E)$, which surjects to every E_x (in the proof above the space V is spanned by ψ_i times a basis section of $E|_{U_i}$).

2.5 Proof in the equivariant case: one has two construct a finite dimensional subspace
$$V \subset W := C(X; E)$$
of the space of the global sections, such that

◦ at every point $x \in X$ the restriction $V \to E_x$ is a surjection,

◦ V is G-invariant.

• This is possible due to a version of Peter-Weyl theorem: ♠ *Suppose W is a topological vector space which is Hausdorff, complete and locally convex. Suppose G (a compact group) acts continuously. Then*
$$W_a = \{v \in W \;\;:\;\; \dim(span\ Gv) < \infty\}$$
is dense in W. [Mos61, 2.16]

◦ (The subscript a in the notation evokes to the Peter-Weyl classical statement, that polynomial functions are dense in all continuous functions $C(G) = C(G; \mathbb{C})$.)

• By the theorem above, at each point $x \in X$ we can choose sections spanning the fiber E_x and belonging to W_a. By compactness of X we can choose finitely many sections spanning fibers at each point. □

2.6 Proof of (♠): if $X = G$ and $E = X \times \mathbb{C}$, then this is the Peter-Weyl theorem:

$$C(G)_a \quad \text{is dense in} \quad C(G)$$

with the norm sup. Suppose $w \in W$, $\phi \in C(G)$. Let

$$w_\phi = \int_G \phi(g) g(w)\, dg \qquad \text{(Haar measure)}\,.$$

Let $U \subset W$ be a convex neighbourhood of 0 in V. We will construct an element $w_\phi \in W_a$, such that $w - w_\phi \in 2U$. Let $V \subset G$ be a neighbourhood of e, such that $g(w) - w \in U$ for $g \in V$. Let $\psi : G \to \mathbb{R}_+$, $supp(\psi) \subset V$, $\int_G \psi(g) dg = 1$. Then

$$w_\psi - w = \int_G (\psi(g) g(w) - w) dg = \int_G \psi(g)(g(w) - w) dg \in U$$

by convexity. The map

$$C(G) \to W\,,$$

$$\phi \mapsto w_\phi$$

is continuous. Chose $\phi \in C(G)_a$, such that

$$w_\psi - w_\phi = \int_G (\psi(g) - \phi(g)) g(w) \in U$$

by density of $C(G)_a \subset C(G)$. Hence

$$w - w_\phi = w - w_\psi + w_\psi - w_\phi \in 2U\,.$$

The vector w_ϕ is contained in a finite dimensional representation, since for $h \in G$

$$h(w_\phi) = h \int_G \phi(g) g(w) dg = \int_G \phi(g) h g(w) dg = \int_G \phi(g h^{-1}) g(w) dg = w_{h\phi}\,.$$

But $h\phi \in span\{g\phi \;:\; g \in G\} = span\{\phi_1, \phi_2, \dots, \phi_m\}$ which is a finite dimensional space by assumption on ϕ. By the definition of w_ϕ (linearity)

$$w_{h\phi} \in span\{w_{\phi_1}, w_{\phi_2}, \dots, w_{\phi_m}\}\,.$$

□

2.7 Corollary: Every G-equivariant vector bundle is a subbundle of a trivial bundle $V_X = X \times V$, where V is a representation of G and the action of G on $X \times V$ is diagonal.

- For every G bundle E there exist a G-bundle F, such that $E \oplus F \simeq V_X$.
- Every G-bundle of rank n is isomorphic to the pull-back (TBA) from the tautological bundle over $\mathrm{Gr}_n(V)$ for an appropriate G-representation V.
- Allowing V to be of infinite dimension it is possible to fix an universal space V. It is necessary to assume that any representation of G is a direct summand of V infinitely many times.

Pull-back

2.8 For $f : X \to Y$ and a vector bundle $p : E \to Y$ let f^*E (also denoted by $f^!E$) is defined by

$$f^*E = E \times_Y X = \{(v, x) \in E \times X \;:\; p(v) = f(x)\,\}\,.$$

- We have $(f^*E)_x \simeq E_{f(x)}$.

2.9 Theorem: *If $f, g : X \to Y$ are homotopic, then $f^*E \simeq g^*E$.*
(I copy the proof from [AB64]. Assumption: X is a compact Hausdorf space.)

2.10 Lemma: *Suppose E, F are vector bundles over X and let $Y \subset X$ be a closed subspace, such that $E|_Y \simeq F|_Y$. Then there exists an open set $U \supset Y$, such that $E|_U \simeq F|_U$.*

• Consider the bundle $\mathrm{Hom}(E, F)$. The isomorphism over Y defines a section of $\mathrm{Hom}(E, F)|_Y$. This section can be extended on each open set of the trivializing cover, and patched together using partition of unity. We obtain a global section $\phi : X \to \mathrm{Hom}(E, F)$. The condition ,,$\phi_x : E_x \to F_x$ is an isomorphism" is an open condition. □

• In the equivariant case the same proof works, , we can extend nonequivariantly and after that average over the group G - which is compact by assumption.

2.11 Lemma: *Let $E \to X \times [0, 1]$ be a vector bundle, $\iota_t : X \to X \times [0, 1]$ the inclusion $x \mapsto (x, t)$, Then $\iota_t^* E \simeq \iota_s^* E$ for all $s, t \in [0, 1]$.*

• Let $F = p^* \iota_t^* E$, where and $p : X \times [0, 1] \to X$ is the projection. Over $X_t := X \times \{t\}$ we have an isomorphism $E|_{X_t} \simeq F|_{X_t}$. Thus $E|_{X_s} \simeq F|_{X_s}$ for s sufficiently close to t. This means that $E|_{X_s} \simeq E|_{X_t}$. Using compactness of $[0, 1]$ and transitivity of the relation $\simeq$ we obtain the claim of the lemma. □

• Theorem 2.9 follows from Lemma in the standard way.

• For the equivariant case we have to assume that the homotopy is G-invariant.

3 K-theory

3.1 The set $Vect_G(X) = \bigsqcup_{n=0}^{\infty} Vect_G^n(X)$, whose elements are the isomorphism classes of complex G-vector bundles over X is semigroup.

• The above definition applies to connected X. If X is not connected, we allow that the ranks of the vector bundles vary:

$$K_G(X) = \bigoplus_{Y \text{ component of X/G}} Vect_G(\pi^{-1}Y)\,,$$

where $\pi : X \to X/G$.

3.2 Segal's K-theory for compact groups

$$K_G(X) = Groth(Vect_G(X))$$

is the associated Grothendieck group.

• The elements of $K_G(X)$ are formal differences $[E] - [F]$ and

$$[E] - [F] = [E'] - [F'] \quad \Leftrightarrow \exists Z \in Vect_G(X) \quad E' \oplus F \oplus Z = F' \oplus E \oplus Z\,.$$

Later we will skip the brackets in the notation.

• The additional summand is necessary to have transitivity of the relation $\sim$. Note that in $Vect_G(X)$ there is no cancellation property

$$A \oplus Z \simeq B \oplus Z \qquad \not\Rightarrow \qquad A \simeq B$$

3.3 Tensor product defines the ring structure in $K_G(X)$. The unit is $\mathbb{1}_X$.

3.4 K-groups are modules over $R(G) = K_G(\star)$.

3.5 Note that if there is an exact sequence of G-vector bundles

$$0 \to E_1 \to E_2 \to E_3 \to 0$$

then in $K_G(X)$

$$E_1 + E_3 = E_2$$

because in the topological category every exact sequence splits over the base which is paracompact.

• One can always find a hermitian product in E_2 and $E_3 \simeq E_1^{\perp}$,

3.6 The above statement is not true in the category of holomorphic bundles, thus in the algebraic K-theory we impose the relation

$$0 \to E_1 \to E_2 \to E_3 \to 0 \qquad \Rightarrow \qquad [E_1] + [E_3] = [E_2]$$

• Very general construction: Suppose $\mathcal{C}$ is an additive category with a distinguished class of ,,exact sequences". (We do not assume that $\mathcal{C}$ is abelian, but the class of ,,exact sequences" has to satisfy certain set of natural axioms, eg it is an additive subcategory of an abelian category, see [Wei13, §II.7].)

- Examples:

∘ All coherent sheaves $\mathcal{C} = Coh(X)$, where X is an algebraic variety.

∘ Locally free sheaves $\mathcal{C} \subset Coh(X)$, where X is an algebraic variety.

In both cases the ,,exact sequences" are just exact sequences in the abelian category $Coh(X)$

∘ Theorem: if X is smooth then the resulting K-theories are isomorphic

$$K_{G,loc.free}(X) \xrightarrow{\simeq} K_{G,coh}(X)$$

We will discuss only the topological K-theory. See [Seg68].

3.7 For a representation V let $V_X = X \times V$. By (2.7) every element of $K_G(X)$ is represented by $E - V_X$.

3.8 We say that vector bundles E, F are stably isomorphic, if there exist G-representations V, W, such that $E \oplus V_X \simeq F \oplus W_X$.

• The stable isomorphism classes is an abelian group with respect to $\oplus$. The opposite bundle exists by (2.7).

- Let $p : X \to \star$ be the map to the point. Define

$$\widetilde{K}_G(X) := coker(p^* : K_G(\star) \to K_G(X)) .$$

• The image of p^* consiss of the virtual bundles $V_X - V'_X$. Therefore the cokernel can be identified with classes of stable isomorphism classes of G-bundles.

3.9 $K_G(X)$ as a functor:

$$K_G : \mathbf{hTop}_G^{op} \longrightarrow R(G) - \text{algebras} .$$

3.10 Extreme examples:

- If $X = G/H$ then $K_G(X) = R(H)$
- If $X = G \overset{H}{\times} Y$, for a H-space Y, then $K_G(X) = K_H(Y)$
- If G acts freely on X then $\pi^* : K(X/G) \to K_G(X)$ is an isomorphism, with inverse:

$$(E \to X) \mapsto (E/G \to X/G) ,$$

$$E \simeq E/G \times_{X/G} X.$$

- More generally: if $N \triangleleft G$ acts freely on X, than $K_{G/N}(X/N) \simeq K_G(X)$.
- If G-action on X is trivial then

$$K_G(X) \simeq K(X) \otimes R(G) .$$

∘ This is due to the functorial isomorphism in representation category of compact groups: Let $\{V_i\}_{i \in I}$ be the set of all simple objects. Then the natural map

$$\bigoplus_{i \in I} V_i \otimes \mathrm{Hom}_G(V_i, V) \xrightarrow{\simeq} V$$

is an isomorphism.

∘ It follows that any bundle over $X = X^G$ is isomorphic to

$$\bigoplus_{i\in I} V_i \otimes \mathrm{Hom}_G(V_i, E)\,.$$

Here the action of G on the second factor is trivial.

∘ The subbundle, E_i, the image of $V_i \otimes \mathrm{Hom}_G(V_i, E)$, is a generalization of the eigen-subbundle for $G = T \simeq (S^1)^r$. Note that E_i might be of higher rank than $\dim V_i$.

• Exercise: Suppose $E \to X$ is as a trivial bundle $E \simeq X \times \mathbb{C}^n$ as a nonequivariant vector bundle. Describe all possible structures of a G-vector bundle.

3.11 With the fixed point $\star$ we have a canonical splitting

$$K_G(\star) \overset{p^*}{\to} K_G(X) \overset{\iota^*}{\to} K_G(\star)\,,$$

thus

$$K_G(X) \simeq K_G(\star) \oplus \widetilde{K}_G(X)\,, \quad \widetilde{K}_G(X) \simeq ker(\iota^*)\,.$$

• If X has no fixed point, then $K_G(\star) \to K_G(X)$ does not have to be a monomorphism, e.g. for $X = G$.

3.12 Let us define

$$K_G(X, A) := \widetilde{K}_G(X/A)\,.$$

By homotopy invariance

$$K_G(X, A) = \widetilde{K}_G(X \cup CA)\,,$$

where $X \cup CA$ has the distinguished point $A \times \{0\} \cup \star \times [0, 1]$ shrunk to one point.

• Note that if A is a G-contractible subspace of a compact G-space X, then

$$K_G(X, A) \simeq \widetilde{K}_G(X)\,.$$

3.13 Descrtiption of $K_G(X, A)$: the elements are represented by the formal differences $E - F$, a with a given isomorphism

$$\phi : E|_A \to F|_A\,.$$

These generators are subjects to a stable isomorphism relation.

• Let $\pi : X \to X/A$. Given a stable isomorphism class E on X/A. Let $V = E|_\star$. Define the pair of bundles

$$\widetilde{E} = \pi^* E, \quad \widetilde{F} = V_X\,,$$

then clearly we have an ismorphism $\widetilde{E}_A \simeq V_A$ on A.

• Conversely, suppose $E|_A \simeq F|_A$. Adding a bundle to both sides we can assume that $F = V_X$, and we have an isomorphism $E|_A \simeq V_A$. By (2.10) we can assume that the isomorphism extends to a neighbourhood $U \supset A$. Therefore we can define a bundle E' on X/A, since we have a trivialization on the neighbourhood $U/A \supset \star$.

3.14 Continuity: Suppose there is a directed system of G-invariant closed subset

$$X \supset \cdots \supset A_i \supset A_{i+1} \supset \ldots\,, \qquad \bigcap_{i=1}^{\infty} A_i = A\,.$$

3.15 Theorem: *The natural map*

$$\varinjlim_i K_G(A_i) \longrightarrow K_G(A)$$

is an isomorphism.

• Injective: suppose $E|_A \simeq F|_A$, then $E|_U \simeq F|_U$ for some neighbourhood of A. Since X is compact, $A_i \subset U$ for i large enough and for such i we have $[E|_{A_i}] = [F|_{A_i}]$.

- Surjective: Given a bundle $E \to A$. We can assume that $E \subset V_A$ for some representation. It is given by the image of an idempotent $p \in End(V_A)$. The space of idempotents in $End(V_X)$ forms a (trivial) fibration over X. It is possible to extend the section from A to a neighbourhood of A. One should do it equivariantly with respect to the G action.
- Segal provides a concrete recipe: we extend p anyhow, however equivariantly (possibly averaging) and we apply the transformation

$$P \mapsto \Phi(P) = \tfrac{1}{2\pi i}\int_\gamma (zI - P)^{-1} dz\,,$$

where γ is a circle around 1 with radious 1/2. It is assumed, that P has no eigenvalues in that circle.

  ◦ Exercise: $\Phi(P)$ is an idempotent.

3.16 Segal's convention: for noncompact spaces $K_G(X) := \widetilde{K}_G(X^c)$, where X^c is the one point compactification (denoted in [Seg68] by X^+ and should not be confused with adding one point to a compact space).

Extending $K_G(-)$ to a cohomology theory

3.17 We work with the G-spaces with a distinguished fixed point, fixed by G. Let CX be the reduced cone, $SX := S^1 \wedge X \simeq CX \times_X CX \simeq CX/X$ be the reduced suspension.

3.18 Proposition: The sequence (with the natural maps)

$$\widetilde{K}_G(X \cup_A CA) \longrightarrow \widetilde{K}_G(X) \longrightarrow K_G(A)$$

is exact.

- $A \to X \cup_A CA$ is null-homotopic, so the composition is 0.
- If $E|_A = 0$, then it means that is that $E|_A \oplus V_A \simeq V'_A$. This allows to define a bundle, which is $E \oplus V_X$ over X and V'_X over CA.

3.19 Puppe sequence. For $f : A \to X$ denote by Cf the mapping cone $CA \cup_f X$. We have an embedding $j : X \to Cf$ and homotopy retraction $Cj \overset{\simeq}{\to} SA$. Moreover the cone of $k : Cf \to SA$, is homotopy equivalent to SX.

- We have the long exact sequence

$$\ldots \longrightarrow \widetilde{K}_G(SX) \longrightarrow \widetilde{K}_G(SA) \longrightarrow \widetilde{K}_G(C(f)) \longrightarrow \widetilde{K}_G(X) \longrightarrow \widetilde{K}_G(A)\,.$$

3.20 Definition. For $q \geq 0$ let

$$\widetilde{K}_G^{-q}(X) = \widetilde{K}_G(S^q X)\,.$$

- To have a cohomology theory it is missing to define $\widetilde{K}^q_G(X)$, $q > 0$.

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