Automorphisms of fusion systems and p-completed classifying spaces of finite simple groups

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Fix a prime p. The fusion system of a finite group G with respect to a Sylow subgroup $S \in \mathrm{Syl}_p(G)$ is the category $\mathcal{F}_S(G)$ whose objects are the subgroups of S, and whose morphisms are the homomorphisms induced by conjugation in G. We describe results which compare, when G is simple, the automorphism groups $\mathrm{Out}(G)$, $\mathrm{Out}(\mathcal{F}_S(G))$, and the group $\mathrm{Out}(BG^\wedge_p)$ of homotopy classes of self homotopy equivalences of the space $BG^\wedge_p$. For example, when G is simple and S is nonabelian, and G is A_n for $n \equiv 0, 1 \pmod{p}$, a group of Lie type in defining characteristic p, or a sporadic group, then $\mathrm{Out}(G) \cong \mathrm{Out}(BG^\wedge_p)$ with just four exceptions. We discuss this result, and also the remaining (more complicated) case when G is simple of Lie type in characteristic different from p.

This comparison of $\mathrm{Out}(BG^\wedge_p)$ with $\mathrm{Out}(G)$ is important when trying to determine whether an extension of fusion systems is realized by an extension of groups.