Algebry i grupy Liego

12.3.2026

http://www.mimuw.edu.pl/%7Eaweber/

1 Examples first

References for this lecture:

• Parts of [Kirillov, An Introduction to Lie Groups and Lie Algebras] §2

1.1 Let G be a topolpgical T_1 space with a continuous map

$$m : G \times G \to G\,, \qquad \nu : G \to G$$

satisfying the axiom of a group
– the multiplication μ,
– taking the inverse ν.

• In other words G is a ,,group object" in the category of topological spaces.

1.2 Examples
- discrete groups
- $\mathbb{K}_+$, $\mathbb{K}^*$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ (quaternions)
- compact torus $(S^1)^r$
- complex torus $(\mathbb{C}^*)^r$
- S^3 as a subgroup of $\mathbb{H}^*$
- $U(n)$, $SU(n)$ subgroups of $GL_n(\mathbb{C})$, $SL_n(\mathbb{C})$
- $O(n)$, $SO(n)$ subgroups of $GL_n(\mathbb{R})$, $SL_n(\mathbb{R})$
- $Sp(n)$ the subgroup of $GL_n(\mathbb{H})$ preserving the norm $|v|^2 = \sum_{i=1}^n |v_i|^2$
- matrix groups preserving a given quadratic form (or other structure, e.g. the octonionic multiplication)
- $O(m,n)$, the subgroup of $GL_{m+n}(\mathbb{R})$ preserving a nondegenerate symmetric form of the type (m,n).
- groups of isometries of a compact Riemannian manifold (can be realized as a matrix group)
- Heisenberg group N/Z where

$$N = \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}, \qquad Z = \begin{pmatrix} 1 & 0 & \mathbb{Z} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

cannot be realized as a matrix group

1.3 Exercise: $U(n)$, $SU(n)$, $SO(n)$, $Sp(n)$ are connected, $O(n)$ has two components

1.4 Exercise: $\pi_1(U(n)) = \mathbb{Z}$, $\pi_1(SU(n)) = 1$, $\pi_1(SO(n)) = \mathbb{Z}_2$ for $n \geq 3$ (long exact sequence of homotopy groups needed)

1.5 Exercise: Elements of $Sp(n)$ preserve the form $\mathbb{H}^n \times \mathbb{H}^n \to \mathbb{H}$ given by $(v,w) = \sum_{i=1}^n v_i \overline{w}_i$.

1.6 Two approaches to Lie groups
- study of compact Lie groups
- study of complex algebraic reductive groups (definition later)

1.7 Noncompact or nonreductive groups are more difficult; theory of nilpotent or solvable groups is a separate subject.

1.8 But any connected Lie group G contains a maximal compact subgroup K (which is unique up to a conjugation) and as a C^∞-manifold $G \simeq K \times \mathbb{R}^n$. (Cartan-Iwasawa-Malcev Theorem)

1.9 For every connected Lie group we have a decomposition (as a topological space)

$$G = K \times A \times N$$

where K is maximal compact, $A \simeq \mathbb{R}^k$, N is a nilpotent group, $\simeq \mathbb{R}^\ell$ as a topological space. This is Iwasawa decomposition. The special case is the Gram-Schmidt orthogonalization process

$$GL_n(\mathbb{R}) = O(n) \times (\mathbb{R}_{>0})^n \times N\,,$$

$$GL_n^+(\mathbb{R}) = SO(n) \times (\mathbb{R}_{>0})^n \times N\,,$$

where N consist of the upper-triangular matrices with 1's at the diagonal.

1.10 Every compact Lie group can be embedded into $U(n)$ as a closed subgroup.

1.11 Classification of compact connected groups [Cartan]: every such G is of the form $\widetilde{G}/A$, where A is a finite abelian group and $\widetilde{G} = \prod_{i=1}^k H_i$ and H_i is a torus $(S^1)^r$ or a simple[1] simply-connected compact group, which is of the form

- $SU(n)$ (Type A_{n-1})

  ◦ $X \in M_{n\times n}(\mathbb{C})$, $\det(X) = 1$, $\overline{X}^T X = I$

- $\widetilde{SO(n)} = Spin(n)$ (Type B_m for $n = 2m + 1$ or Type D_n for $n = 2m$). Here $\widetilde{SO(n)}$ means the two-fold cover. (Spin group.)

  ◦ $X \in M_{n\times n}(\mathbb{R})$, $\det(X) = 1$, $X^T X = I$

- $Sp(n)$ (Type C_n — the compact symplectic group)

  ◦ $X \in M_{n\times n}(\mathbb{H})$, $\overline{X}^T X = I$

- Exceptional groups E_6, E_7, E_8, G_2 or F_4

1.12 Definitions of compact simple Lie groups have common pattern, while the field varies

- $\mathbb{C}$ – Type A_n (preserving hermitian product)

- $\mathbb{R}$ – Type B_n and D_n (preserving scalar product)

[1] Simple Lie group means that the every proper normal subgroups is finite.

- $\mathbb{H}$ – Type C_n (preserving scalar product in the quaternionic space)

- octonions $\mathbb{O}$ are related to exceptional groups, e.g. $G_2 = Aut(\mathbb{O})$ preserving scalar product

1.13 Empirical Fact: For each compact Lie group G there exists a complex Lie group $G_{\mathbb{C}}$, the complexification of G, in which G is the maximal compact subgroup. The group $G_{\mathbb{C}}$ is defined by a polynomial formula in $GL_N(\mathbb{C})$ for some N

Compact Lie Group	**Corresponding Complex Algebraic Group**
$U(1) = S^1$ (circle)	$GL(1, \mathbb{C}) = \mathbb{C}^*$ (multiplicative group of nonzero complex numbers)
$U(n)$	$GL(n, \mathbb{C})$
$SU(n)$	$SL(n, \mathbb{C})$
$O(n)$	$O(n, \mathbb{C})$
$SO(n)$	$SO(n, \mathbb{C})$
$Sp(n)$	$Sp(n, \mathbb{C})$
G_2	$G_2(\mathbb{C})$
F_4	$F_4(\mathbb{C})$
E_6	$E_6(\mathbb{C})$
E_7	$E_7(\mathbb{C})$
E_8	$E_8(\mathbb{C})$

1.14 Simple, simply-connected groups:

- $SL(n, \mathbb{C})$ (Type A_{n-1})

- $\widetilde{SO_n(\mathbb{C})} = Spin_n(\mathbb{C})$ (Type B_m for $n = 2m + 1$ or Type D_n for $n - 2m$), where $SO_n(\mathbb{C})$ is a subgroup of $SL_n(\mathbb{C})$ preserving a fixed nondegenerate symmetric form.

- $Sp_n(\mathbb{C})$ (Type C_n), where $Sp_n(\mathbb{C})$ is a subgroup of $GL_{2n}(\mathbb{C})$ preserving a fixed nondegenerate antisymmetric form.

- Complex exceptional group of the type E_6, E_7, E_8, G_2 or F_4, eg. $(G_2)_{\mathbb{C}} \subset GL_7(\mathbb{C})$ is the group preserving certain exterior 3-form.

1.15 Exercise: The real symplectic group $Sp_n(\mathbb{R}) \subset GL_{2n}(\mathbb{R})$ (appears in real symplectic geometry or in classical mechanics) is noncompact and its maximal compact subgroup is equal to $U(n)$.

Topological properties (to be verified on tutorials)

1.16 Suppose $H \leq G$ is a subgroup. Then the action $G \times G/G \to G/H$ is continuous.

1.17 Theorem. Suppose $H \leq G$ is a subgroup. Then

- the quotient $\pi : G \to G/H$ map is open:

1.18 If H is closed, then the space G/H is regular [points are closed and every closed subset F of G/H and a point not contained in F admit non-overlapping open neighborhoods; hence G/H is Hausdorff].

1.19 Corollary: Any topological group is a regular topological space.

1.20 Theorem: Let G_0 be the connected component of $1 \in G$. (It is the biggest set, which is connected and contains 1.) Then G_0 is a closed normal subgroup.

1.21 If G_0 is open, then G/G_0 is a discrete group.

1.22 If G is a connected topological group, then any open neighbourhood of the identity generates the group.

1.23 A normal discrete subrgroup of a connected group lies in its center.

Lie groups

From now on we assume that G is a C^∞-manifold, μ, ν are smooth. Homomorphism of Lie groups are assumed to be smooth maps.

1.24 By a Lie subgroup of G we understand $H \subset G$ which is both a subgroup and a closed submanifold.

• In fact, if H is a closed subgroup, then necessirly it has to be a submanifold, but the proof is not easy.

1.25 Example: $G = GL_n(\mathbb{C})$, $H = SL_n(\mathbb{C})$ is given by the equation $\det A = 1$.

1.26 The map

$$\exp : M_{n\times n}(\mathbb{C}) \to GL_n(\mathbb{C})$$
$$X \mapsto I + X + \frac{1}{2!}X^2 + \frac{1}{3!}X^3 + \dots$$

maps a neighbourhood of 0 diffeomorphically to a neighbourhood of I. (see [Kir. Th. 2.29])

• An inverse in the neighbourhood of 0 is given by:

$$\log(1+x) = x - \tfrac{x}{2} + \tfrac{x^2}{2} - \dots$$

• $\exp(X) \in SL_n(\mathbb{C})$ iff and $\exp(tr(X)) = 1$, thus (assuming that X is small) this means that $tr(X) = 0$. This shows that the tangent space to $SL_n(\mathbb{C})$ is the space of traceless matrices

$$\mathfrak{sl}_n(\mathbb{C}) = \{X \in M_{n\times n}(\mathbb{C}) \ : \ tr(X) = 0\}\,.$$

We use Gothic letters to denote the tangent spaces at e, thus to agree with that convention we write $\mathfrak{gl}_n(\mathbb{C}) = M_{n\times n}(\mathbb{C})$ for the space of all matrices.

1.27 If $[X,Y] = 0$, then $\exp(X)\exp(Y) = \exp(Y)\exp(X) = \exp(X+Y)$, but in general there is no equality.

• There are formulas for multiplication of matrices in the coordinates on $GL_n(\mathbb{K})$ given by logarithm, so called Baker–Campbell–Hausdorff formula, see [Kir §3.7].

• Exercise: check $\exp(x)\exp(y) = \exp(z)$, where

$$z = x + y + \tfrac{1}{2}[x,y] + \tfrac{1}{12}[X,[X,Y]] - \tfrac{1}{12}[Y,[Y,X]] + \dots$$

1.28 The group G acts on itself. We have natural actions:

• Left translation $L_g : G \to G$, $h \mapsto gh$

• Right translation $R_g : G \to G$, $h \mapsto hg^{-1}$

• Adjoint action $Ad_g : G \to G$, $h \mapsto ghg^{-1}$

1.29 Ad_g preserves $e = 1$, thus the derivative, also denoted by Ad_g acts on $\mathfrak{g} = T_1G$.

1.30 Example: $G = GL_n(\mathbb{K})$, is an open set in the space of matrices, thus $T_1GL_n(\mathbb{K}) = M_{n\times n}(\mathbb{K}) =: \mathfrak{gl}_n(\mathbb{K})$. The action on $X \in \mathfrak{gl}_n(\mathbb{K})$ is given by $Ad_g(X) = gXg^{-1}$.

• Proof. Let $\gamma(t) = \exp(tX)$.

$$Ad_g(X) = \frac{d}{dt}\Big(g\gamma(t)g^{-1}\Big)_{|t=0} = \frac{d}{dt}\Big(g(1 + tX + \tfrac{t^2}{2}X^2 + \dots)g^{-1}\Big)_{|t=0} =$$

$$= \frac{d}{dt}\Big(1 + tgXg^{-1} + \tfrac{t^2}{2}gX^2g^{-1} + \dots\Big)_{|t=0} = gXg^{-1}.$$

1.31 Tangent to adjoint: The map $Ad : GL_n(\mathbb{K}) \to GL(\mathfrak{gl}_n(\mathbb{K}))$ has its derivative at $1 \in GL_n(\mathbb{K})$ denoted by ad:

$$ad : \mathfrak{gl}_n(\mathbb{K}) \to \mathrm{End}(\mathfrak{gl}_n(\mathbb{K}))\,.$$

• The value $ad(X) \in \mathrm{End}(\mathfrak{gl}_n(\mathbb{K}))$ is denoted by ad_X.

• The value $ad(X)(Y) = \mathfrak{gl}_n(\mathbb{K})$ is denoted by ad_XY.

• Let us compute:

$$ad_X(Y) = \frac{d}{dt}\Big(\exp(tX)Y\exp(-tX)\Big)|_{t=0} =$$

$$= \Big(X\exp(tX)Y\exp(-tX) + \exp(tX)Y(-X)\exp(-tX)\Big)|_{t=0} = XY - YX = [X, Y]$$

•

$$\boxed{ad_XY \to [X, Y]}$$

1.32 The above computation works well for any subgroup of $GL_n(\mathbb{C})$ and can be generalized to an abstract Lie group.

2 From Lie groups to Lie algebras

2.1 Definition: A map of Lie groups $f : G_1 \to G_2$ is a morphism of Lie groups if it is a C^∞-map and a homomorphism of groups. Examples:

• the quotient map $\mathbb{R} \to \mathbb{R}/\mathbb{Z} = S^1$

• the quotient map $\mathbb{R}^n \to \mathbb{R}^n/\Lambda \simeq (S^1)^n$, where $\Lambda \simeq \mathbb{Z}^n$ is a lattice.

• $SU(2) \to SO(3)$ (discussed during problem sessions)

• $\exp : \mathbb{C} \to \mathbb{C}^*$

The above examples have discrete kernels and are surjective (they are coverings of the underlying topological spaces)

• $\det : GL_n(\mathbb{K}) \to \mathbb{K}^*$ (for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$)

• $GL_n(\mathbb{K}) = GL(\mathbb{K}^n) \to GL(\wedge^k\mathbb{K}^n) \simeq GL_{\binom{n}{k}}(\mathbb{K})$

2.2 Theorem [Kir. Cor 2.10]: Let $f : G_1 \to G_2$ be a morphism of Lie groups, such that the map of the tangent spaces at e:

$$f_* : T_eG_1 \to T_eG_2$$

is surjective and G_2 is connected, then f is surjective.

• Proof: the image contains a nieghbourhood of e.

2.3 [Kir.Th 2.11] If G is a Lie group and H a closed Lie subgroup, then G/H has a structure of a manifold and the quotient map $\pi : G \to G/H$ is a smooth fibration.

• If H is normal, then G/H is a Lie group.

2.4 [Kir. Th 2.15] If $f : G_1 \to G_2$ is a morphism of Lie groups, then $ker(f)$ is a Lie subgroup, the induced map $G_1/ker(f) \to G_2$ is a immersion, but the image is not necessarily a Lie subgroup (e.g. the image $\mathbb{R} \to (S^1)^2$ it can be dense).

2.5 For $X \in \mathfrak{gl}_n(\mathbb{K})$ the map

$$\mathbb{K} \to GL_n(\mathbb{K})$$

$$t \mapsto e^{tX}$$

is a morphism of Lie groups.

• In general $e^{X+Y} \neq e^X e^Y$ unless $XY = YX$. That is $\exp : \mathfrak{g} \to G$ is not a group homomorphism.

2.6 The commutator $[X, Y]$ and the map exp e^X makes sens for all the matrix groups, i.e. Lie subgroups of $GL_n(\mathbb{K})$.

Group	**Definition**	**Lie Algebra**
General Linear $\mathrm{GL}(n, \mathbb{K})$	$\{A \in M_{n\times n}(\mathbb{K}) \mid \det(A) \neq 0\}$	$\mathfrak{gl}(n, \mathbb{K})$: all $n \times n$ matrices
Special Linear $\mathrm{SL}(n, \mathbb{K})$	$\{A \in M_{n\times n}(\mathbb{K}) \mid \det(A) = 1\}$	$\mathfrak{sl}(n, \mathbb{K}) = \{A \mid \mathrm{tr}(A) = 0\}$
Orthogonal $\mathrm{O}(n, \mathbb{K})$	$\{A \in M_{n\times n}(\mathbb{K}) \mid A^T A = I\}$	$\mathfrak{o}(n, \mathbb{K}) = \{A \mid A^T + A = 0\}$
Special Orthogonal $\mathrm{SO}(n, \mathbb{K})$	$\mathrm{O}(n, \mathbb{K}) \cap \mathrm{SL}(n, \mathbb{K})$	$\mathfrak{so}(n, \mathbb{K}) = \mathfrak{o}(n, \mathbb{K})$
Unitary $\mathrm{U}(n)$	$\{A \in M_{n\times n}(\mathbb{C}) \mid \overline{A}^T A = I\}$	$\mathfrak{u}(n) = \{A \mid \overline{A}^T + A = 0\}$
Special Unitary$\mathrm{SU}(n)$	$\mathrm{U}(n) \cap \mathrm{SL}(n, \mathbb{C})$	$\mathfrak{su}(n) = \{A \in \mathfrak{u}(n) \mid \mathrm{tr}(A) = 0\}$
Symplectic $\mathrm{Sp}(n, \mathbb{K})$	$\{A \in M_{2n\times 2n}(\mathbb{C}) \mid A^T J A = J\}$	$\mathfrak{sp}(n, \mathbb{K}) = \{A \mid A^T J + JA = 0\}$
Compact Symplectic $\mathrm{Sp}(n)$	$U(2n) \cap Sp(n, \mathbb{C})$	$\mathfrak{sp}(n) = \mathfrak{u}(2n) \cap \mathfrak{sp}(n, \mathbb{C})$

Table 1: Classical Groups and Their Lie Algebras, see [Kir. Tab 2.1-3]

• One can check that the commutator operations preserves $\mathfrak{g} \subset \mathfrak{gl}_n(\mathbb{C})$, also exp maps $\mathfrak{g}$ to G, not only on the level on tangent spaces. This is not a coincidence, it follows from an abstract approach to Lie groups.

General groups

Abstract manifolds having group operations, without specified embedding to the matrix group, which might not exist at all.

2.7 The map exp can be generalized to arbitrary groups. [Kir §3] We proceed as follows:

• The left translation allows to identify $\mathfrak{g}$ with left invariant vector fields on G. For $\lambda \in \mathfrak{g}$

$$X_\lambda(g) = (L_g)_*(\lambda) \quad \text{here } (L_g)_* : \mathfrak{g} = T_1 G \overset{\simeq}{\to} T_g G$$

$$(L_g)_*(X_\lambda)(h) = (L_g)_*(X_\lambda(g^{-1}h)) = (L_g)_*(L_{g^{-1}h})_*(\lambda) = (L_h)_*(\lambda) = X_\lambda(h) \in T_h G$$

• Having a vector field X_λ we find the integral curve $\gamma_\lambda : \mathbb{R} \to G$, such that $\gamma_\lambda(0) = 1$.

∘ In the case of the matrix groups $\gamma_\lambda(t) = \exp(t\lambda)$

∘ For arbitrary groups we prove that γ_λ is indeed defined for all $t \in \mathbb{R}$ and we define $\exp(\lambda) := \gamma_\lambda(1)$.

∘ Note $\exp(t\lambda) := \gamma_\lambda(t)$

2.8 Theorem (compare [Kir. Prop 3.1]): The map $t \mapsto \gamma_\lambda(t)$ is a group homomorphism.

• Proof. It follows from the properties of flows generated by a field vector. Suppose $\phi_t : G \to G$ is the flow of X_λ. Then

$$\phi_{s+t}(g) = \phi_s(\phi_t(g))\,.$$

Note that for any $g, h \in G$:

$$L_g(\phi_t(h)) = \phi_t(L_g(h)) \quad \text{i.e.} \quad g\phi_t(h) = \phi_t(gh)$$

hence

$$\exp((s+t)\lambda) = \phi_{s+t}(e) = \phi_s(\phi_t(e)) = \phi_s(\exp(t\lambda)) = \exp(t\lambda)\phi_s(e) = \exp(t\lambda)\exp(s\lambda)\,.$$

2.9 There are bijections

$$\mathfrak{g} = T_eG \quad \leftrightarrow \quad \text{Left invariant vector fields} \quad \leftrightarrow \quad \text{1–parameter subgroups}$$

By a *1–parameter subgroup* we understand a homomorphism $\mathbb{R} \to G$. It does not have to be injective.

2.10 Reminder from differential geometry [Kir §3.5]: commutator of vector fields: a vector field X on a manifold M defines a differential operator on function $D_X : C^\infty(M) \to C^\infty(M)$ of the first order[2] The commutator $[D_X, D_Y]$ is again a differential operator of the first order.

• Exercise: If $M = \mathbb{R}^2$, $X = (X_1, X_2)$, $Y = (Y_1, Y_2)$, then $Z = (Z_1, Z_2)$:

$$Z_1 = X_1\frac{\partial}{\partial x}Y_1 + X_2\frac{\partial}{\partial y}Y_1 - Y_1\frac{\partial}{\partial x}X_1 - Y_2\frac{\partial}{\partial y}X_1$$

$$Z_2 = X_1\frac{\partial}{\partial x}Y_2 + X_2\frac{\partial}{\partial y}Y_2 - Y_1\frac{\partial}{\partial x}X_2 - Y_2\frac{\partial}{\partial y}X_2$$

2.11 In general:

$$([\mathbf{X}, \mathbf{Y}])_i = \sum_j X_j\frac{\partial Y_i}{\partial x_j} - Y_j\frac{\partial X_i}{\partial x_j}$$

2.12 The commutator of left-invariant vector fields is a left invariant vector field.

• This way we obtain a bilinear operation $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ denoted by $[X, Y]$.

• The commutator of vector fields (as the commutator of any linear operators) satisfies the Jacobi identity

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0\,.$$

[2] The first order differential operator is a linear map defined on $C^\infty(M)$ satisfying $D(fg) = D(f)g + fD(g)$, i.e. the Leibniz rule.

2.13 Abstract Lie Algebra. Let $\mathbb{K}$ be a field. An *abstract Lie algebra* over $\mathbb{K}$ is a vector space $\mathfrak{g}$ over $\mathbb{K}$ equipped with a bilinear map called the *Lie bracket*:

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g},$$

satisfying the following axioms:

1. **Bilinearity:** For all $X, Y, Z \in \mathfrak{g}$ and $a, b \in \mathbb{K}$,

$$[aX + bY, Z] = a[X, Z] + b[Y, Z], \quad [Z, aX + bY] = a[Z, X] + b[Z, Y].$$

2. **Antisymmetry:** For all $X, Y \in \mathfrak{g}$,

$$[X, Y] = -[Y, X].$$

   In particular (if $char(\mathbb{K}) \neq 2$), this implies that $[X, X] = 0$ for all $X \in \mathfrak{g}$.

3. **Jacobi Identity:** For all $X, Y, Z \in \mathfrak{g}$,

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

2.14 The Jacobi identity can be rewritten as the Leibniz rule: let $ad_X(Y) = [X, Y]$, then the Jacobi identity is equivalent to

$$ad_X([Y, Z]) = [ad_X Y, Z] + [X, ad_Y Z]\,.$$

2.15 Ado theorem states that every finite-dimensional Lie algebra over a field $\mathbb{K}$ of characteristic zero can be realized as a subalgebra of $\mathfrak{gl}_n(\mathbb{K})$ for some n.

• It is not true that every Lie group can be realized as a subgroup of $GL_n(\mathbb{R})$ (e.g. the Heisenberg group).

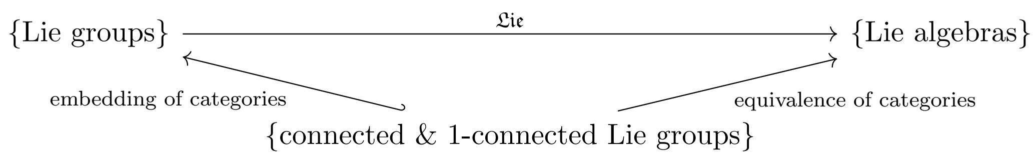
2.16 The correspondence $\mathfrak{Lie} : G \rightsquigarrow \mathfrak{g} = \mathfrak{Lie}(G)$ extends to a functor: Lie groups $\to$ Lie algebras extends to a functor, i.e. any morphism of Lie algebras $f : G_1 \to G_2$ induces

$$\mathfrak{Lie}(f) = f_* : \mathfrak{Lie}(G_1) \to \mathfrak{Lie}(g_2)\,,$$

such that

$$(f \circ g)_* = f_* \circ g_*\,.$$

2.17 The main theorem of Lie theory, [Kir §3.8]: The functor $\mathfrak{Lie}$ restricted to the subcategory of connected, simply connected[3] Lie groups is an equivalence of categories:



• Difficult part:

[3] i.e. $\pi_1(G)$ is trivial.

∘ 1) For any Lie algebra $\mathfrak{g}$ there exist a group G, such that $\mathfrak{Lie}(G) = \mathfrak{g}$

∘ 2) For any map of Lie algebras $f : \mathfrak{g}_1 = \mathfrak{Lie}(G_1) \to \mathfrak{g}_2 = \mathfrak{Lie}(G_2)$ there exist a map of Lie groups inducing f, provided that $\pi_1(G_2) = 1$.

• The proof is ommited, see a (not quite complete, but illuminating) exposition in [Segal: in *Lectures on Lie Groups and Lie Algebras*, §5]

2.18 (Summary of properties of the abstract exp. [Kir §3.1]) Let X be a left-invariant vector field on the Lie group G, and let $\phi_{X,t} : G \to G$ be the flow of this vector field.

• The flow $\phi_{X,t}$ is defined for all $t \in \mathbb{R}$.

• The map $t \mapsto \phi_{X,t}(e)$ is a homomorphism of Lie groups.

• For $G = GL_n(\mathbb{R})$, we have $\phi_{X,t}(e) = \exp(tA)$, where $A = X(e) \in \mathfrak{gl}_n(\mathbb{R})$.

• If $[X, Y] = 0$, then the flows commute.

2.19 [Kir. Prop. 3.12(3)] In general: The difference between two flows, is approximately related to the Lie bracket $[X, Y]$. Let

$$\gamma(t) = \phi_{X,t} \circ \phi_{Y,t} \circ \phi_{X,-t} \circ \phi_{Y,-t}(e).$$

We have

$$\gamma(0) = e, \quad \dot\gamma(0) = 0, \quad \ddot\gamma(0) = [X, Y]\,.$$

This formula is valid for arbitrary vector fields on differentiable manifolds.

Elements of the proof of the Lie theorem

2.20 The tangent map $D\exp : T_0\mathfrak{g} = \mathfrak{g} \to T_1G = \mathfrak{g}$ is the identity. Therefore the image of exp contains a neighbourhood of 1, hence the image of exp generates the identity component of G.

• Exercise: not every element of $SL_2(\mathbb{R})$ lies in the image of exp.

2.21 The map $\exp : \mathfrak{g} \to G$ commutes with the morphisms of Lie groups.

$$\begin{array}{ccc} \mathfrak{g}_1 = \mathfrak{Lie}(G_1) & \xrightarrow{\varphi_*} & \mathfrak{g}_2 = \mathfrak{Lie}(G_2) \\ \downarrow{\scriptstyle \exp} & & \downarrow{\scriptstyle \exp} \\ G_1 & \xrightarrow{\varphi} & G_2 \end{array}$$

• That is so because $\varphi_* : T_gG_1 \to T_{\phi(g)}G_2$ transports the vector $X_\lambda(g)$ to $X_{\phi_*(\lambda)}(\phi(g))$.

2.22 A morphism of Lie groups induces a map of tangent spaces. This map preserves the structure of the Lie algebra.

• This can be deduced from the interpretation of $[X, Y]$ given in (2.19).

2.23 [Kir 3.12(1)] If $\varphi, \psi : G_1 \to G_2$ are two morphisms of Lie groups and $T_1\phi = T_1\psi$ then $\phi = \psi$ provided that G_1 is connected.

• Proof: the map is determined by the map restricted to the image of exp, which contains a neighbourhood of 1.

2.24 [Kir. Th. 3.41-2] We assume that G_1 is connected. From the above we see that

$$\mathfrak{Lie} : \mathrm{Hom}_{\text{Lie groups}}(G_1, G_2) \to \mathrm{Hom}_{\text{Lie algebras}}(\mathfrak{g}_1, \mathfrak{g}_2)$$

is injective. In general it is not surjective, e.g. for $G_1 = S^1$, $G_2 = \mathbb{R}$. The Lie theorem says that for a given map of Lie algebras $\mathfrak{g}_1 \to \mathfrak{g}_2$ there exists a map from a covering of G_1. Let $\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_2$ and $\mathfrak{h} = graph(\mathfrak{g}_1 \to g_2) \subset \mathfrak{g}$. Translating $\mathfrak{h}$ to any point of G we obtain a distribution. One has to show, that this distribution is integrable, i.e. it is tangent to a foliation. Taking the leaf passing through 1 we obtain $\widetilde{G}_1$, a covering of the original G_1.

2.25 [Kir Th. 3.40] The fact that $\mathfrak{Lie}$ is essentially surjective on objects follows from Ado theorem (every Lie algebra is realized as a subalgebra of matrices). Having $\mathfrak{g} \subset \mathfrak{gl}_n(\mathbb{R})$ we construct G as before.

3 Repsentations of Lie groups [Kir. §4]

3.1 Our strategy: to understand Lie groups we study the action of G on $\mathfrak{g}$

$$Ad : G \to GL(\mathfrak{g})\,.$$

• Need to develop general theory of actions of groups on vector spaces.

3.2 [Kir §4.1] A representation of G on a vector space V is a homomorphism of Lie groups $\rho : G \to GL(V)$. That is for $g \in G$ we have a linear map $\rho(g) : V \to V$ such that

$$\rho(gh) = \rho(g)\rho(h)\,.$$

Often we write gv instead of $\rho(g)(v)$ for $v \in V$.

3.3 Representations of G (over a fixed $\mathbb{K}$) form a category, denoted by $Rep(G)$ or G-Mod (i.e. G-modules).

• Morphisms in $Rep(G)$ are denoted by

$$\mathrm{Hom}_G(V, W) = \Big\{\phi : V \to W \;\;:\;\; f \text{ is linear and } \forall g \in G \;\; g\phi(v) = \phi(gv)\Big\}.$$

3.4 Similarly we have representation of Lie algebras and a functor $Rep(G) \to \mathfrak{g}$-Mod.

$$\rho : \mathfrak{g} \to \mathrm{End}(V) \quad \text{linear}\,,$$

$$\rho([X, Y]) = \rho[X]\rho[Y] - \rho[Y]\rho[X]\,.$$

• Morphism of representations of Lie algebras:

$$\mathrm{Hom}_{\mathfrak{g}}(V, W) = \Big\{\phi : V \to W \;\;:\;\; f \text{ is linear and } \forall X \in \mathfrak{g} \;\; X\phi(v) = \phi(Xv)\Big\}.$$

3.5 Examples of G-representations:

• $\mathbb{1} = \mathbb{C}$ the trivial representation: for all $g \in G$, $v \in \mathbb{C}$ we have $gv = v$.

• G with the action by Ad – the adjoint representation

• Natural (or defining) representation: for the groups defined as subgroups of $\mathrm{GL}_n(\mathbb{C})$. E.g. the group $SU(n)$ naturally acts on $\mathbb{C}^n$.

3.6 Example of $\mathfrak{g}$ representations:

• $\mathbb{1} = \mathbb{C}$ the trivial representation: for all $X \in \mathfrak{g}$, $v \in \mathbb{C}$ we have $Xv = 0$.

• Adjoint representations

$$ad : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}), \quad ad_X(Y) = [X, Y]\,.$$

○ The formula

$$ad_{[X,Y]}Z = ad_X ad_Y Z - as_Y ad_X Z$$

is equivalent to the Jacobi identity.

• Natural (or defining) representation: for the groups defined as subalgebras of $\mathfrak{gl}_n(\mathbb{C})$. E.g. the group $\mathfrak{su}_n(\mathbb{C}) = \{A \in \mathfrak{gl}_n(C) \mid A + \overline{A}^T = 0,\ \ tr(A) = 0\}$ naturally acts on $\mathbb{C}^n$.

3.7 Space of invariants. Notation: for a representation $\rho : G \to GL(V)$ let

$$V^G = \{v \in V \ : \ \forall g \in G \ \ \rho(g)(v) = v\}\,.$$

3.8 [Kir §4.2] Operations on representations of a (Lie) group

• direct sum, tensor product

○ $\mathrm{Hom}(V, W)$: for $\phi \in \mathrm{Hom}(V, W)$

$$\big(\rho(g)(\phi)\big)(v) := \rho_W(g)(\phi(\rho_V(g^{-1}(v)))$$

or in short

$$(g\phi)(v) = g\phi(g^{-1}v)\,.$$

Note that $\mathrm{Hom}_G(V, W) = \mathrm{Hom}(V, W)^G$ (the invariants of G action on $\mathrm{Hom}(V, W)$).

• in particular the dual $V^* = \mathrm{Hom}(V, \mathbb{K})$:

$$(g\phi)(v) = \phi(g^{-1}v)$$

• external and symmetric powers (assume characteristic 0)

○ symmetric tensors $Sym^kV \subset V^{\otimes k}$, other presentation $Sym^kV = (V^{\otimes k})^{\Sigma_k}$, where Σ_k is the permutation group acting by permutations of factors

○ anti-symmetric tensors $\wedge^kV \subset V^{\otimes k}$, other presentation $\wedge^kV = (V^{\otimes k})^{\Sigma_k}$. Here the action is by permutation times the sign.

○ Exercise that both $\wedge^kV$ and Sym^kV are isomorphic with $V^{\otimes k}/\Sigma_k$ for appropriate actions of the permutation group Σ_k

• Decomposition of representation

○ $Sym^2V \oplus \wedge^2V = V \otimes V$

○ but $Sym^3V \oplus \wedge^3V \subsetneq V \otimes V \otimes V$

3.9 The corresponding Lie algebra representations $\mathfrak{g} \to \mathfrak{gl}(V) = \mathrm{End}(V)$

• direct sum $X(v, w) = (Xv, Xw)$

• tensor product $X(v \otimes w) = Xv \otimes w + v \otimes Xw$

- for $\phi \in \mathrm{Hom}(V, W)$ we define $X\phi$ by $(X\phi)(v) = \phi(-Xv) + X\phi(v)$
- dual $(X\phi)(v) = \phi(-Xv)$

◦ Exercise $\mathrm{Hom}_{\mathfrak{g}}(V, W) = \{\phi \in \mathrm{Hom}(V, W) \; : \; X\phi = 0\}$.

3.10 Suppose G is connected.

- If G has trivial center, then $G \to Aut(\mathfrak{g}) = GL(\mathfrak{g})$ is injective.
- If G is simple, i.e. does not have connected normal subgroups and it is not abelian, then the kernel of

$$G \to Aut(\mathfrak{g})$$

is finite.

- If $\mathfrak{g} = \mathfrak{Lie}(G)$ and G is simple, then $\mathfrak{g}$ embeds in $\mathrm{End}(\mathfrak{g}) = \mathfrak{gl}(\mathfrak{g})$.

3.11 The center of the Lie algebra is defined by

$$Z(\mathfrak{g}) = \{X \in \mathfrak{g} \; : \; \forall Y \in \mathfrak{g} \;\; [X, Y] = 0\}\,.$$

3.12 Ado's theorem: every Lie algebra can be realized as a subalgebra of $\mathfrak{gl}_n(\mathbb{C})$.

- A very easy case: suppose $Z(\mathfrak{g}) = 0$, then the map $ad : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ is injective.

3.13 Category of representatrions. Let G be a fixed Lie group, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The objects of the representation category $Rep_{\mathbb{K}}(G)$ (for short $Rep(G)$ when $\mathbb{K}$ is fixed, most often $\mathbb{C}$). There are kernels, cokernels, direct sums etc. Also, there is a tensor product. The main task is to understand that category. It turns out that for compact groups the category is *semisimple*, i.e. every object is a sum of simple objects. On the other hand the decomposition of tensor products into simple objects is a problem, which has solution expressed in the combinatorial terms for classical groups.

- Important question: for which groups all the representations are completely irreducible?

3.14 [Kir §4.3] A representation is irreducible (or simple) if it is nonzero and it does not contain proper subrepresentations.

3.15 A representation is called completely reducible (semisimple) if it is a direct sum of irreducible repesentations.

3.16 Schur Lemma [Kir §4.4, Lemma 4.23]: Let V, W be irreducible representation and $\phi \in \mathrm{Hom}(V, W)$, then either

- $\phi = 0$
- or ϕ is an isomorphism (the $V \simeq W$).

3.17 Suppose V is irreducible

- if $\mathbb{K} = \mathbb{C}$, then $\mathrm{End}_G(V) = \mathrm{Hom}_G(V, V) \simeq \mathbb{C}$.
- if $\mathbb{K} = \mathbb{R}$, then $\mathrm{End}_G(V)$ is a division algebra (every nonzero element is invertible), hence isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

3.18 [Kir Cor. 4.25] Suppose V is completely irreducible, let $V \simeq \bigoplus_i V_i^{n_i}$ be a decomposition into irreducibe summands. There is a canonical isomorphism:

$$\bigoplus_i V_i \otimes \mathrm{Hom}_G(V_i, V) \to V$$

induced by the bilinear maps

$$V_i \times \mathrm{Hom}_G(V_i, V) \to V, \quad (v, \phi) \mapsto \phi(v).$$

• The number $n_i = \dim \mathrm{Hom}_G(V_i, V)$, so it does not depend on the choices made.

• The image of $V_i \otimes \mathrm{Hom}_G(V_i, V)$ is called the isotipical summand.

•

$$\mathrm{End}_G(V) = \bigoplus_i GL_{n_i}(\mathbb{C}) \,.$$

3.19 Theorem: If G is compact, then any representation decomposes into a sum of irreducible representations. (TBC)

• The proof employs an invariant Hermitian inner product, constructed through the Haar measure. (TBC)

3.20 The groups for which every representation decomposes into a sum of irreducible representations are called reductive.

• For complex groups we consider only holomorphic representations.

3.21 Examples of nonreductive groups: $\mathbb{C}$, the group of upper-triangular matrices.

3.22 Examples of reductive groups:

• Compact groups.

• $\mathbb{C}^*$, $GL_n(\mathbb{C})$, $SL_n(\mathbb{C})$, $Sp_n(\mathbb{C})$, $O_n(\mathbb{C})$, $SO_n(\mathbb{C})$, ...

• It turns out that the complex reductive groups are exactly the complexifications of compact groups.