Isomorphic–free generation of some classes of triangulations without repetitions *

[Extended Abstract]

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ABSTRACT
We present an algorithm to generate all non–isomorphic disconnectible polyhedra without repetitions. Then we discuss a generalisation of this algorithm for the following class of simplicial polyhedra: 4–connected simplicial polyhedra.

Keywords
Triangulation, graph isomorphism, group automorphism

1. GENERAL DESCRIPTION
This work contributes to the research direction, which concerns enumeration and generation of various types of objects (graphs, polyhedra, triangulations etc.) We are able to generate explicitly non-isomorphic simplicial polyhedra of the following two types: disconnectible polyhedra and 4–connected simplicial polyhedra - without repetitions. They are also special types of the triangulations of the 2-sphere. No control on isomorphism is required, and there is only a very restricted check on automorphism. Our approach can be called “voluminous”, because we deal with tetrahedra or pentahedra as generating blocks. We came across this approach while trying to construct a reasonable initial 2D (closed) triangulation from scattered 3D data. Our method might also be useful in simulations of growing processes of certain structures (molecular structures, corals, fractals). There are several algorithms to enumerate and/or generate non-isomorphic triangulations of the 2-sphere (or some classes of those triangulations), but it seems there is yet no one that completely avoids the check on isomorphism [1, 2, 3]. One of the first recursive methods for generating triangulations of the 2-sphere was developed by Robert Bowen and Stephen Fisk in 1967 [2]. They developed an algorithm for constructing all (non isomorphic) triangulations of the 2-sphere with \( N \) vertices from those with \( N - 1 \) vertices. Let \( T \) be a triangulation with \( N \) vertices, \( E \) edges, \( F \) faces. It is easy to see that there is only one triangulation with 4 vertices and one with 5. Let us denote the degree of vertex \( V \) (its valency) by \( d(v) \) and the number of vertices in \( T \) with this degree by \( V_{d(v)} \). By using the Euler formula:

\[
N - E + F = 2
\]

and the following evident equation:

\[
\sum_{v \in N} V_{d(v)} d(v) = 2E;
\]

we get:

\[
\sum_{v \in N} V_{d(v)} (6 - d(v)) = 12
\]

From the last equation we can easily deduce the following statement: for any triangulation of the 2-sphere there is at least one vertex with degree three, four or five, because the left side of the equation must be positive. On the basis of this statement Bowen and Fisk introduced three operations whose applications to all possible triangulations with \( N - 1 \) vertices yield all triangulations with \( N \) vertices. These operations consist of inserting a new vertex of degree three, four or five only to an appropriate triangulation with \( N - 1 \) vertices. Namely, a vertex of degree three can be added to all triangulations, of degree four - only to triangulations with the property that the minimal valency of all vertices is equal or larger than 4, and a vertex with degree 5 - only to triangulations with the property that the minimal valency of all its vertices is equal to 5. We have to add the vertex of degree 5 in such a way that the minimal valency 5 of all the vertices is kept. Therefore, we can distinguish the following three operations:

- Operation 1. Adding a new vertex of degree three. We add a new vertex by connecting it to the three vertices of the same face.
- Operation 2. Adding a new vertex of degree four. We add a new vertex by connecting it first to the three vertices of some face as in the previous case, and then...
connecting it to the one of the three remaining vertices of the adjacent faces. The internal edge, which is formed as a result of this procedure, is deleted.

- Operation 3. Adding a new vertex of degree five. First we implement Operation 2 (for appropriate triangles) and then connect a new vertex to the fifth vertex, which is one of the four remaining vertices of the adjacent faces. The minimal valency of the vertices may not decrease. The formed internal edge is deleted.

We must keep, of course, in mind, that those faces of the previous triangulation to whose vertices new vertices were connected, are not more the faces of the new triangulation.

The procedure to construct all non-isomorphic triangulations with \( N \) vertices then is the following: first apply the three operations in an appropriate way to all triangulations with \( N - 1 \) vertices and then check the obtained triangulations on isomorphism.

In our approach we prefer to avoid any check on isomorphism. For any triangulation with \( N \) vertices the corresponding graph is determined, and on the base of this graph we can define the groups of automorphisms of the triangulation. We simplify this procedure by introducing the labelling of vertices in a special way.

To a triangulation \( T \) of the 2-D sphere corresponds a certain polyhedron \( P \). It is easy to see that the insertion of a vertex of degree three in \( T \) is equivalent to the procedure of adding a new tetrahedron to \( P \), which has precisely an exterior triangle in common with \( P \).

As we deal with non-isomorphic triangulations, the positions of vertices have no role, and we can deform our triangles as we like. Hence, we can form from two adjacent triangles a square, and from three - a pentagon. Consequently, the insertion of a vertex of degree 4 in \( T \) is the same as the addition to \( P \) a 4-pyramid with four triangles as the lateral faces and a square as the base (a pentahedron). The base of the pyramid is "glued" to two adjacent triangles in \( P \). The adjacent edge of these two triangles (the "extra" edge) is deleted. The insertion of a vertex of degree 5 is the same as the addition to \( P \) a 5-pyramid with five lateral faces - triangles and the base as a pentagon (a hexahedron). Two "extra" edges are deleted.

By adding a tetrahedron, a 4-pyramid and a 5-pyramid in a different way and in a different order, one can obtain all non-isomorphic polyhedra (triangulations of the 2-sphere). A tetrahedron is "glued" to a polyhedron with \( N - 1 \) vertices along a triangle, which we call a 3-cut. Consequently a 4-pyramid is "glued" along a 4-cut, and a 5-pyramid - along a 5-cut. Hence, a polyhedron can be constructed via a sequence of 3-, 4-, and 5-cuts. Unfortunately, we were unable to avoid the check on isomorphism.

We changed our strategy and decided to try to generate triangulations (polyhedra) of three different classes: those that can be generated only from tetrahedra, those - from 4-pyramids, and those from 5-pyramids.

2. REFERENCES

